极化簇的一致赋值稳定性

Uniformly valuative stability of polarized varieties

（申请清华大学理学博士学位论文）

培 养 单 位： 数学科学系

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二〇二三年十一月

Uniformly valuative stability of polarized varieties

Dissertation submitted to

Tsinghua University

in partial fulfillment of the requirement

for the degree of

Doctor of Philosophy

in

Mathematics

by

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November, 2023

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摘 要

在复流形上寻找典则度量是一个自然的问题，可追溯到 Calabi。对于极化流形，常数量曲率 Kähler 度量是一个好的选择。从上世纪 80 年代开始，Yau, 田, Donaldson 和其他数学家指出典则度量的存在性等价于一个代数几何的条件，所谓的 K-稳定性。这个问题在总体上仍然是开的。近年来，在 Fano 的情形下，这些问题有了很大的进展。陈-Donaldson-孙和田独立地证明了 Fano 流形的 K-多稳定性推出正 Kähler-Einstein 度量的存在性。

在代数理论里，人们从双有理几何的角度来研究 Fano 簇的 K-稳定性。Fujita 和李驰用赋值重新解释了 Fano 簇的 K-稳定性。这既是所谓的 Fujita-李判据。Fano 簇 K-稳定性的一个基本完善的理论被建立。而 Fujita-李判据在这个理论里起着至关重要的作用。因此发展极化簇 K-稳定性的赋值判据是必不可少的。基于此，Dervan-Legendre 首先考虑了极化簇的赋值稳定性并且证明了 K-稳定性和赋值稳定性的部分等价。

关于一致稳定性的一个基本问题是在极化线丛的小扰动下它是否还保持。这个问题源于 LeBrun-Simanca 的一个经典结果，他们建立了常数量曲率度量在扰动下的开性。我们对一致赋值稳定性的上述问题给一个肯定的回答，并且证明射影簇的丰沛锥里满足一致赋值稳定性的线丛是一个开子锥。我们的赋值稳定性的定义是强于 Dervan-Legendre 的定义。我们也定义一致赋值稳定性阈值，这推广了 Fujita-Odaka 的 δ-不变量，并且我们证明这个不变量的连续性。

作为赋值稳定性的应用，我们对 Donaldson 的 J-方程研究赋值 J-稳定性，并且证明 J-方程赋值判据的一个方向。另外，我们得到一个极化环簇的体积的上界。而且我们的体积上界不需要任何底流形的 Ricci 曲率的假设。

关键词：K-稳定性；赋值稳定性；β-不变量；赋值 J-稳定性；体积上界

Abstract

Finding the canonical metrics on complex manifolds is a natural problem, dating back to Calabi. The constant scalar curvature Kähler (cscK) metric is a good candidate in the case of polarized manifold. Since the 80s, Yau, Tian, Donaldson and others proposed the existence of the canonical metric is equivalent to an algebro-geometric notion, the so-called K-stability. The problem is widely open in general. There have been considerable strides on these ideas for the Fano case in recent years. Chen-Donaldson-Sun and Tian independently proved that K-polystability of Fano manifolds implies the existence of positive Kähler-Einstein metrics.

In the algebraic side, Fujita and Li re-interpreted K-stability of Fano varieties in terms of valuations. This is the so-called *Fujita-Li criterion*. People study K-stability of Fano varieties from the viewpoint of birational geometry. An almost complete theory of K-stability of Fano varieties is established. The Fujita-Li criterion for K-stability of Fano varieties has played an essential role in this theory. Thus developing the valuative criterion of K-stability of polarized varieties is necessary. Dervan-Legendre first considered the valuative stability of polarized varieties and showed a partial equivalence.

A basic question about uniform stability is whether it is preserved under small perturbations of the polarization or not. This question is motivated by a classical result of LeBrun-Simanca, in which they established openness results for perturbations of cscK metrics. We give an affirmative answer to the above question for uniformly valuative stability and show that the uniformly valuative stability locus is an open subcone of the ample cone of projective varieties. Our definition is stronger than that of Dervan-Legendre. We also define a uniformly valuative stability threshold, which generalizes the δ-invariant of Fujita-Odaka, and prove the continuity of this invariant.

As applications of valuative stability, we study the valuative J-stability for Donaldson's J-equation and show a direction of valuative criterion of J-equation. In addition, we obtain an upper bound of the volume of polarized toric variety. Our upper bound does not need any assumption about the Ricci curvature of underlying manifolds.

Keywords: K-stability; Valuative stability; β-invariant; Valuative J-stability; Upper bound of the volume

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List of Symbols and Acronyms

t	The coordinate of $\mathbb{R}$
τ	The coordinate of $\mathbb{C}$
r	The parameter of test curves
ω	The fixed Kähler metric
ϕ	The function in the Archimedean side
$\mathcal{H}$	The space of Kähler potential in the class $[\omega]$
Psh	ω-plurisubharmonic functions
$\mathcal{E}^1$	ω-plurisubharmonic functions of the finite energy
$\omega_\phi := \omega + \mathrm{dd^c}\phi$	The Kähler form of ϕ
Φ	The plurisubharmonic path in Psh or the corresponding function on $X \times \mathbb{C}$
φ	The function in the non-Archimedean side
$(\mathcal{X}, \mathcal{L})$	The test configuration of (X, L)

We work throughout over the complex number $\mathbb{C}$. A *variety* is always assumed to be a connected, reduced, separated and of finite type scheme over Spec $\mathbb{C}$. Unless we say specifically, in this thesis, fix all divisors as Cartier divisors. For the convenience of writing, we do not distinguish between divisors and line bundles.

Chapter 1 Introduction

1.1 CscK problem

Finding the canonical metric on Kähler manifolds is central problem in Kähler geometry. The first result of this form is the classical uniformization theorem in dimension 1. In higher dimension, Kähler-Einstein metrics, constant scalar curvature (cscK for short) metrics, extremal Kähler metrics are good candidates. In this thesis, we focus on the cscK metrics of polarized manifolds. In particular, the Kähler-Einstein metric is the cscK metric.

For Kähler-Einstein metrics, when the first Chern class is negative or zero, Yau [1] (also Aubin [2] in the negative case) showed that the compact Kähler manifold admits a unique Kähler-Einstein metric, solving the famous conjecture of Calabi by using the continuity method.

The case of positive first Chern class is very difficult. Some mathematicians constructed obstructions of existence, for example Matsushima [3], Futaki [4] etc. The obstructions of Matsushima are that Kähler-Einstein manifolds have reductive automorphism group. But there exists some trivial example of Fano manifolds with non-reductive automorphism group. Futaki constructed an integral invariant, the so-called *Futaki invariant*. By definition of Futaki invariant, Kähler-Einstein manifolds have vanishing Futaki invariant.

Yau [5] conjectured that in this case the existence of a Kähler-Einstein metric is related to the stability of the underlying manifold in the sense of Mumford's geometric invariant theory. Tian made great progress towards understanding this (see [6]) by giving an analytic condition which is equivalent to the existence of the Kähler-Einstein metric. This condition is the *properness* of the Ding functional, which is an energy functional on the Kähler class whose critical points are Kähler-Einstein metrics. In [6], Tian also defined K-stability of Fano manifold based on the generalized Futaki invariant of Ding-Tian [7] and conjectured that it is equivalent to the existence of Kähler-Einstein metrics.

In [8], Donaldson showed that the scalar curvature arises as a moment map for a suitable infinite dimensional symplectic action (see also Fujiki [9]), the so-called *Fujiki-Donaldson picture*. By Kempf-Ness Theorem, this explained on a formal level why the existence of a Kähler-Einstein metric, or more generally a cscK metric, is related to the stability of the variety.

In particular, Donaldson [10] generalized Tian's definition of K-stability by giving an algebro-geometric definition of the Futaki invariant, and conjectured that it is equivalent to the existence of a cscK metric. This is the so-called *Yau-Tian-Donaldson* (*YTD for short*) *conjecture*. The notion of K-stability of a polarised variety has played an important role in algebraic geometry, especially Fano varieties, in recent years.

The YTD conjecture is widely open in general. There have been considerable strides on these ideas for the Fano case in recent years. Chen-Donaldson-Sun [11] and Tian [12] independently proved that K-polystability implies the existence of Kähler-Einstein metrics on Fano manifolds, solving this conjecture in the Fano case (also see [13], [14], [15], [16] for other different methods).

Unfortunately, examples in [17] showed that positivity of the Donaldson-Futaki invariant for algebraic test-configurations may not be enough to ensure the existence of a cscK metric. A stronger notion, the so-called *uniform K-stability*, is introduced by the thesis [18] and deeply developed in [19] and [20], which becomes a new candidate for the stability criterion of the existence of a cscK metric. When the automorphism group of a manifold is discrete, the uniform YTD conjecture states that uniform K-stability is equivalent to the existence of a cscK metric. Very recently, Li [21] proved the existence of cscK metrics under the condition of *uniform K-stability for model filtration*, which is stronger than the original uniform K-stability. Moreover, his approach also holds when the automorphism group is non-discrete.

1.2 Valuative stability

In the algebraic side, the theory has achieved substantial progress. The main breakthrough is due to Fujita [22] (also Fujita and Odaka [23]) and Li [24], which re-interprets K-stability in terms of valuations by the algebraic invariant, the so-called *δ-invariant* [23] or *β-invariant* [22], [24] (see Section 5.1). One can test K-stability of a Fano variety by computing its δ-invariant or β-invariant. This is the so-called *Fujita-Li criterion* (see Theorem 5.1).

People study K-stability of Fano varieties from the viewpoint of birational geometry. An almost complete theory of K-stability of Fano varieties is established. From this powerful theory, one can construct a desirable moduli space of K-stable Fano varieties, the so-called *K-moduli space*. There are many important works along these lines, due to Xu, Liu, Zhuang, Blum, etc. (see [25], [26], [27], [28], [29], [30], etc.). We refer the reader to an excellent survey [31] for the algebraic theory of K-stability of Fano varieties. Very recently, Liu-Xu-Zhuang [32] proved that the K-moduli space is proper by solving two profound and challenging conjectures, the so-called Higher Rank Finite Generation conjecture and Optimal Destabilization conjecture. As an application of those conjectures, they also show that K-stability is equivalent to uniform K-stability for a log Fano pair with the discrete automorphism group. Moreover, their argument also holds for the non-discrete automorphism group. The Fujita-Li criterion for K-stability of Fano varieties has played an essential role in all of these developments.

To study K-stability of polarised varieties, the next step is to develop the Fujita-Li criterion in the polarised case. The original definition of K-stability involves $\mathbb{C}^*$-degenerations of a polarised variety, the so-called *test configurations*. Donaldson [10] associates a numerical invariant to each test configuration, the so-called *Donaldson-Futaki invariant*. K-stability means that this invariant is always positive. By works of Boucksom, Jonsson, etc. (see [19], [33]), we can identify a test configuration with a finitely generated $\mathbb{Z}$-filtration on the section ring of the polarization.

For any valuation, one can associate a filtration to this valuation. When this filtration is finitely generated, such a valuation is called a *dreamy valuation*. A valuation is called a *divisorial valuation* if it is induced by a prime divisor over the variety (see Chapter 5 for the definition). A divisor is called a *dreamy divisor* if the corresponding divisorial valuation is dreamy. Dervan and Legendre [34] define a new β-invariant for polarised varieties, which generalizes Fujita's original β-invariant, by computing the Donaldson-Futaki invariant of the test configuration associated with a dreamy divisor. They showed that K-stability over integral test configurations is equivalent to valuative stability over dreamy divisors. Here an integral test configuration means that its central fiber is integral. It gives an expectation to establish the Fujita-Li criterion in the polarized case.

1.3 Main results

A basic question about uniform stability is whether it is preserved under small perturbations of the polarization or not. This question is motivated by a classical result of LeBrun-Simanca [35], in which they established openness results for perturbations of cscK metrics. Fujita [36] proved the openness of uniform K-stability for the log canonical and log anti-canonical polarization. Note that Fujita's result requires that the base variety can have bad singularities, the so-called *demi-normal pair* (see [37] or [38]). Zhang [39] proved that the valuative stability threshold (δ-invariant) is continuous on the big cone of Fano manifolds. Thus, the openness of uniformly valuative stability holds for Fano manifolds.

In this thesis, we consider the openness of uniformly valuative stability for general projective varieties. This gives an affirmative answer to the above question for uniformly valuative stability. Note that our definition of uniformly valuative stability is stronger than that given by Dervan and Legendre in [34], see Definition 5.3 and Remark 5.2. Our main theorem is

Theorem 1.1 ([40] see Theorem 5.3). For a normal projective variety X, the uniformly valuative stability locus

$$\mathrm{UVs} := \{[L] \in \mathrm{Amp}(X) \mid (X, L) \text{ is uniformly valuatively stable}\} \tag{1.1}$$

is an open subcone of the ample cone $\mathrm{Amp}(X)$.

Together with LeBrun-Simanca's openness, our result fits the expectation of YTD conjecture.

A main difficulty of Theorem 1.1 is to control the difference of the derivative part in the expression of β-invariant for two nearby ample divisors. It is hard to control the difference for all prime divisors in general. In addition, the log discrepancy has no control generally. By considering the derivative part of β-invariant together with the log discrepancy, we obtain a partial control of β-invariant (see Theorem 6.1), which is enough to show our main theorem.

As an immediate application of the main theorem, we obtain

Corollary 1.1 ([40] see Theorem 6.2). For a normal projective variety X, the uniformly valuative stability threshold

$$\mathrm{Amp}(X) \ni L \mapsto \zeta(L) \in \mathbb{R} \tag{1.2}$$

is continuous on the ample cone $\mathrm{Amp}(X)$ (see Definition 6.2 for $\zeta(L)$).

The invariant ζ is motivated by δ-invariant since $\delta - 1$ can be viewed as the stability threshold (in the sense of Definition 6.2) of the original β-invariant. Corollary 1.1 gives a similar result with Zhang [39] for projective varieties. According to the expression of β-invariant, we do not have a canonical formulation to define its corresponding δ-invariant for polarised varieties. Studying the invariant ζ is a good candidate to test valuative stability.

The definition of cscK metrics does not need a polarization. In [41] and [42], they independently defined K-stability for the transcendental Kähler classes. It is natural to extend the valuative stability to any Kähler class of compact Kähler manifolds (see Definition 7.1).

Due to some well-known results about analytic geometry, it is straightforward to see that our argument for the algebraic class can also work for the Kähler class on projective manifolds. We state it as follows,

Theorem 1.2 ([40] see Theorem 7.3). For a projective manifold X, the uniformly valuative stability locus

$$\widehat{\mathrm{UVs}} := \{\alpha \in \mathcal{K} \mid (X, \alpha) \text{ is uniformly valuatively stable}\} \tag{1.3}$$

is an open subcone of the Kähler cone $\mathcal{K}$.

Another topic of this thesis is the J-equation, which is introduced by Donaldson [43] from the point-view of moment maps, as well as by Chen [44] in the study of the Mabuchi K-energy whose critical point is the cscK metric. To state the equation, let (X, ω) be a compact Kähler manifold of dimension n, and let χ be another Kähler metric on X which is not related to ω. The J-equation is the elliptic equation

$$\mathrm{tr}_{\omega_\phi} \chi = c, \tag{1.4}$$

where $\omega_\phi = \omega + \mathrm{dd^c}\phi$ is a Kähler form and c is a constant, only depending on the classes of $[\omega]$ and $[\chi]$,

$$\int_X n\chi \wedge \omega^{n-1} = c \int_X \omega^n. \tag{1.5}$$

The J-equation can be written as the critical point of a functional on the space of Kähler potentials, denoted by $\mathcal{J}_\chi$. When $\chi = -\mathrm{Ric}(\omega)$, then the functional $\mathcal{J}_\chi$ is the energy part of K-energy. Thus, the J-equation appears naturally in the study of the cscK problem.

Based on the fact that the J-equation has a moment map description, Lejmi and Székelyhidi [45] introduced the analogies of K-stability and slope stability for the J-equation, the so-called *J-stability* and *slope J-stability* (see Definition 7.2).

There are many interesting works for the J-equation including [45], [46], [47], [48], [48] etc. Recently, Chen [49] showed the equivalence between the existence of the J-equation and uniform J-stability. He also proved a equivalent numerical condition for the existence of the solution of the J-equation, which is conjectured in [45].

The numerical condition seems to be difficult to check since it involves all analytic subvarieties of the underlying Kähler manifold. Thus, we ask whether there exists an equivalent condition involving just subvarieties of codimension 1 to test the solvability of the J-equation. Motivated by Fujita-Li criterion, we study the J-stability in terms of divisorial valuations to test the existence of J-equations. This seems to be more natural from the view point of birational geometry.

Parallelling the Fujita's β-invariant, We define a invariant $j_H(\cdot)$ (see Section 7.2). Thus, we also define the J-stability threshold, denoted by γ_H (see Section 7.2), similar with δ-invariant. We expect that the γ_H invariant will play a similar role to δ-invariant. We can show a direction of the valuative criterion.

Proposition 1.1 (see Proposition 7.1). If the polarized manifold (X, L) has a unique solution of the J-equation (7.18), then (X, L) is uniformly valuatively J-stable, i.e.,

$$\gamma_H(L) > c. \tag{1.6}$$

Finally, we give an application of valuative stability about the upper bound of the volume of polarized varieties.

The upper bound of volumes of Kähler-Einstein Fano manifolds is given by [50] under a condition about the $\mathbb{C}^*$-action of underlying manifolds. Their approach is analytic and based on the Moser-Trudinger type inequalities. Fujita [51] removed the Berman-Berndtsson's condition and showed the same upper bound of volumes for Kähler-Einstein manifolds by using the purely algebraic method in terms of valuative stability. Later, Zhang [52] showed the same bound of volumes replaced the Kähler-Einstein condition by that the Ricci curvature has positive lower bound. The author of [53] generalized Fujita's result to singular Fano varieties.

As an application of valuative stability of polarized varieties, we obtain an upper bound of the volume of polarized toric varieties. It is well-known that the polarized toric variety is determined by the lattice polytope P_L

$$P_L = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geqslant -a_\rho \text{ for all } \rho \in \Sigma(1)\}. \tag{1.7}$$

Theorem 1.3 (see Theorem 7.6). Let (X, L) be a polarized toric variety of dimension n. Assume (X, L) is K-semistable. Then we have

$$\sqrt[n]{\mathrm{Vol}(L)} \leqslant \max_\rho a_\rho (1 + \frac{n}{n+1} \mu(L) \tau_L(F)) \tag{1.8}$$

where F is an exceptional divisor of blowup at a smooth point. See Section 5.2 for the definitions of $\mu(L)$ and $\tau_L(F)$.

Note that our theorem does not need any assumption about the Ricci curvature of underlying manifolds. It generalizes the upper bound of the volume of [50] in the toric case.

To facilitate access to the individual topics, the sections are rendered as self-contained as possible.

The thesis is organized as follows.

- In Chapter 2, we introduce some background about Kähler geometry and pluripotential theory.
- In Chapter 3, we recall some classical theory about the volume of big divisors and the positive intersection product.
- In Chapter 4, we review the definition of K-stability and non-Archimedean pluripotential theory, and introduce the connection between them.
- In Chapter 5, we compare the valuative stability of Fano varieties and polarized varities.
- In Chapter 6, we give the proof of Theorem 1.1 and Corollary 1.1.
- In Chapter 7, we provide some applications of valuative stability, including valuative stability for the transcendental Kähler class, valuative J-stability and the upper bound of the volume of polarized toric varieties.

Chapter 2 Kähler geometry and pluripotential theory

2.1 Kähler geometry

In this section, we review some basic definitions and notations in Kähler geometry. We assume that the readers are familiar with the notion of complex manifolds.

Let (X, J) be a complex manifold, namely, X is an even dimensional manifold and J is a integrable complex structure on X.

Definition 2.1. A Riemannian metric g on X is called *Hermitian* if $g(JX, JY) = g(X, Y)$ for any tangent vectors X, Y.

Given a Hermitian metric g, one defines

$$\omega(X, Y) := g(JX, Y), \qquad \text{for any } X, Y. \tag{2.1}$$

One can check that ω is antisymmetric in X, Y. Hence, ω defines a real $(1,1)$-form.

Definition 2.2. A Hermitian metric g is called *Kähler* if the associated $(1,1)$-form ω is closed. A tuple (X, g) is called a *Kähler manifold* if X is a complex manifold and g is a Kähler metric.

We always do not distinguish ω and g.

Example 2.1. The complex projective space $\mathbb{P}^n$ has a natural Kähler metric ω_{FS} called the *Fubini-Study metric*, given by

$$\omega_{\mathrm{FS}} := \frac{1}{2}\sqrt{-1}\partial\bar{\partial}\log(|X_0|^2 + \cdots + |X_n|^2), \tag{2.2}$$

where $[X_0 : \cdots : X_n]$ is the homogeneous coordinate of $\mathbb{P}^n$. It is easy to check that ω_{FS} is well-defined.

Example 2.2. A polarized manifold (X, L) consists of a complex manifold together with a ample line bundle L. The polarized manifold is an important class of examples of compact Kähler manifolds. We always denote

$$d_m := \dim H^0(X, L^m). \tag{2.3}$$

Take a basis $\{s_i\}_{i=0}^{d_m-1}$ of $H^0(X, L^m)$ for the large enough $m > 0$, one can define a holomorphic map

$$\begin{aligned} \iota_m : X &\to \mathbb{P}^{d_m-1}. \\ x &\mapsto [s_0(x) : \cdots : s_{d_m-1}(x)] \end{aligned} \tag{2.4}$$

In fact, the map ι_m is a embedding, called *Kodaira embedding*. Then $\frac{1}{m}\iota_m^*\omega_{\mathrm{FS}} \in c_1(L)$ is a Kähler metric.

We recall the $\mathrm{dd^c}$-lemma.

Lemma 2.1 ($\mathrm{dd^c}$-lemma). Let X be a compact Kähler manifold. If ω and η are two real $(1,1)$-forms in the same cohomology class, then there exists a function $f : X \to \mathbb{R}$ such that

$$\eta - \omega = \mathrm{dd^c} f, \tag{2.5}$$

where $\mathrm{dd^c} := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$.

We fix a compact Kähler manifold (X, J, g) of complex dimension $n \geqslant 1$. Let ω be the corresponding Kähler form and ∇ be the Levi-Civita connection of g on TX. We also denote the natural extension of ∇ on $T^{\mathbb{C}}X$ by ∇. Take the holomorphic local coordinate $(z^1, \cdots, z^n)$ of X, the non-zero Christoffel symbols with respect to the basis $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z^i}}\}$ of $T^{\mathbb{C}}X$ are

$$\Gamma^i_{jk} = g^{i\bar{l}}\frac{\partial}{\partial z^j}g_{k\bar{l}}, \quad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \overline{\Gamma^i_{jk}}, \tag{2.6}$$

where $(g^{i\bar{l}})$ is the inverse matrix of $(g_{i\bar{l}})$. The Riemannian curvature components are

$$R_{ijk\bar{l}} = -R_{ij\bar{l}k} = \overline{R_{\bar{i}\bar{j}\bar{k}l}} = -\overline{R_{\bar{i}\bar{j}l\bar{k}}} = 0, \tag{2.7}$$

and

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2}{\partial z^i \partial \overline{z^j}}g_{k\bar{l}} - g^{p\bar{q}}\frac{\partial}{\partial z^i}g_{k\bar{q}}\frac{\partial}{\partial \overline{z^j}}g_{p\bar{l}}. \tag{2.8}$$

The *Ricci curvature* $R_{i\bar{j}}$ is defined by

$$R_{i\bar{j}} := R_{i\bar{j}\ k}^{\ \ k} := -g^{k\bar{l}}R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z^i \partial \overline{z^j}}\log\det(g_{k\bar{l}}). \tag{2.9}$$

The *Ricci form* is defined by

$$\mathrm{Ric}(\omega) := \frac{\sqrt{-1}}{2\pi}R_{i\bar{j}}dz^i \wedge d\overline{z^j} = -\mathrm{dd^c}\log\omega^n. \tag{2.10}$$

A Kähler metric ω is *Kähler-Einstein* (KE for short) if it satisfies

$$\mathrm{Ric}(\omega) = \lambda\omega, \tag{2.11}$$

for some $\lambda \in \mathbb{R}$. By rescaling the metric, we can assume that $\lambda = 1, 0, -1$, which corresponds to the three cases

$$c_1(X) > 0, \quad c_1(X) = 0, \quad c_1(X) < 0. \tag{2.12}$$

When $c_1(X) < 0$, the equation (2.11) was already solved by Aubin [2] and Yau [1]. When

$c_1(X) = 0$, the equation (2.11) always has a unique solution due to Yau [1]. This is the famous Calabi-Yau theorem. When $c_1(X) > 0$ (called *Fano manifold*), the equation (2.11) does not necessarily have a solution. There exists some well-known obstructions, for example Mutsushima's reductiveness [3], Futaki invariant [4], etc.

We set

$$\mathfrak{h} := \left\{ \text{holomorphic vector field } v \text{ such that } v^j = g^{j\bar{k}} \frac{\partial}{\partial \overline{z^k}} f \text{ for some } f : X \to \mathbb{C} \right\}. \tag{2.13}$$

The space $\mathfrak{h}$ is a Lie algebra and is independent of the choice of metrics in the Kähler class $[\omega]$. Mutsushima [3] proved that $\mathfrak{h}$ is reductive provided a Kähler-Einstein Fano manifold.

The *scalar curvature* is defined by

$$\mathrm{Scal}(\omega) := g^{i\bar{j}} R_{i\bar{j}}. \tag{2.14}$$

Definition 2.3. A Kähler metric ω is called a *constant scalar curvature Kähler* (cscK for short) metric if

$$\mathrm{Scal}(\omega) = const, \tag{2.15}$$

which satisfies the following equation

$$g^{i\bar{j}} \partial_i \partial_{\bar{j}} (\log \det(g_{i\bar{j}})) = const \tag{2.16}$$

on compact Kähler manifold.

Note that the constant in (2.16) is the average scalar curvature

$$\widehat{S} = \frac{2n\pi c_1(X) \cup [\omega]^{n-1}}{[\omega]^n}, \tag{2.17}$$

which only depends on X and the Kähler class $[\omega]$. If (X, L) is a polarized manifold and $\omega \in c_1(L)$, then

$$\widehat{S} = n \frac{-K_X \cdot L^{n-1}}{L^n} =: n\mu(L), \tag{2.18}$$

which is a algebraic numerical invariant. We will revisit it in Chapter 5.

In the thesis, we are interested in the existence of cscK metrics.

It is easy to see that the Kähler-Einstein metrics are cscK metrics. Conversely, suppose that X is Fano manifold and $\omega \in c_1(X)$ is a cscK metric, then ω is in fact Kähler-Einstein. Indeed, since both $\mathrm{Ric}(\omega)$ and ω represent $c_1(X)$, then by dd^c-lemma, there exists a real smooth function $f : X \to \mathbb{R}$ such that

$$\mathrm{Ric}(\omega) - \omega = \mathrm{dd^c} f. \tag{2.19}$$

By taking trace of both hand sides in (2.19), it follows that $\mathrm{Scal}(\omega) - n = \Delta f$. But the left hand side is a constant function, which is a harmonic 0-form. By Hodge decomposition theorem, both hand sides must be 0. Thus, f is a constant function. and hence $\mathrm{Ric}(\omega) = \omega$.

The following theorem, due to Futaki [4], gives an obstruction to finding cscK metrics in a Kähler class. It will turn out to be a first glimpse into K-stability.

Theorem 2.1 ([4]). Let (X, ω) be a compact Kähler manifold. one defines the functional $\mathrm{Fut}(v) : \mathfrak{h} \to \mathbb{C}$, called *Futaki invariant*, by

$$\mathrm{Fut}(v) = \int_X f(\mathrm{Scal}(\omega) - \widehat{S})\omega^n, \tag{2.20}$$

where f is a holomorphy potential for v. Then the functional Fut is independent of the choice of metrics in the Kähler class $[\omega]$.

In particular if $[\omega]$ admits a cscK metric, then $\mathrm{Fut} \equiv 0$.

2.2 Pluripotential

In this section, we introduce the pluripotential theory, geodesic rays and the structure of the space of Kähler potentials.

Let (X, ω) be an n-dimensional compact Kähler manifold. We denote $V := \int_X \omega^n$. We refer to [54], [55], [56], for more story of pluripotential theory.

2.2.1 Finite energy potential

Definition 2.4. (i) A function ϕ on the domain $U \subseteq \mathbb{C}^n$ is called *plurisubharmonic* (psh for short) if it is upper semi-continuous and for all complex lines $\Lambda \subseteq \mathbb{C}^n$, the restriction $\phi|_{U\cap\Lambda}$ is subharmonic in $U \cap \Lambda$, i.e.,

$$\phi(x) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \phi(x + e^{\sqrt{-1}\theta}\xi)d\theta, \tag{2.21}$$

for all $x \in U$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(x, \partial U)$.

(ii) A function ϕ is called *ω-psh function* if $\phi \in L^1(X, \mathbb{R} \cup \{-\infty\})$ which can be locally written as the sum of a smooth and a psh function, and such that

$$\omega + \mathrm{dd^c}\phi \geqslant 0 \tag{2.22}$$

in the sense of current.

Denote by $\mathrm{Psh} := \mathrm{Psh}(X, \omega)$ the space of ω-psh functions $\phi : X \to [-\infty, +\infty)$, endowed with L^1-topology as the weak topology. To find canonical metrics in Kähler class $[\omega]$, by $\mathrm{dd^c}$-lemma, we know that it suffices to consider the following space of potential functions

$$\mathcal{H} := \{\phi \in C^\infty(X) | \ \omega_\phi := \omega + dd^c\phi > 0\}, \tag{2.23}$$

called the space of *Kähler potential*. For any $\phi \in \mathcal{H}$, then

$$\omega_\phi^n := (\omega + \mathrm{dd^c}\phi) \wedge \cdots \wedge (\omega + \mathrm{dd^c}\phi) \tag{2.24}$$

defines a positive measure on X, called *the Monge-Ampère measure* of ϕ. Obviously, $\mathcal{H} \subset \mathrm{Psh}$. By Demailly regularization Theorem, every $\phi \in \mathrm{Psh}$ can be written as the point-wise limit of a decreasing sequence of Kähler potentials.

The *Monge-Ampère energy* $E : \mathcal{H} \to \mathbb{R}$ is defined as the antiderivative of the Monge-Ampère measure, given by

$$E(\phi) := \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \phi\omega_\phi^j \wedge \omega^{n-j}, \tag{2.25}$$

for any $\phi \in \mathcal{H}$. It is easy to compute its first order variation

$$\langle E'(\phi), \delta\phi \rangle = \frac{d}{dt}\Big|_{t=0} E(\phi + t\delta\phi) = V^{-1} \int_X \delta\phi\omega_\phi^n, \tag{2.26}$$

for any $\phi \in \mathcal{H}$ and $\delta\phi \in C^\infty(X)$. It follows that

$$E(\phi) - E(\psi) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X (\phi - \psi)\omega_\phi^j \wedge \omega_\psi^{n-j}, \tag{2.27}$$

for any $\phi, \psi \in \mathcal{H}$. In addition, the Monge-Ampère energy E satisfies

$$E(u + c) = E(\phi) + c \text{ for } \phi \in \mathcal{H}, \ c \in \mathbb{R}, \tag{2.28}$$

and

$$\phi \leqslant \psi \Rightarrow E(\phi) \leqslant E(\psi) \text{ for } \phi, \psi \in \mathcal{H}, \text{ with equality iff } \phi = \psi. \tag{2.29}$$

It follows that the functional E admits a unique extension as a monotone, upper semi-continuous functional

$$E : \mathrm{Psh} \to \mathbb{R} \cup \{-\infty\}, \tag{2.30}$$

obtained by setting

$$E(\phi) := \inf\{E(\psi) : \psi \in \mathcal{H}, \psi \geqslant \phi\}, \tag{2.31}$$

for any $\phi \in \mathrm{Psh}$. The space of *finite energy potentials* can be defined as

$$\mathcal{E}^1 = \mathcal{E}^1(M, \omega) := \{\phi \in \mathrm{Psh} : E(\phi) > -\infty\}. \tag{2.32}$$

The convex set

$$\mathcal{E}^1_C(X,\omega) := \left\{ \phi \in \mathcal{E}^1 \,\middle|\, \sup \phi \leqslant C \text{ and } E(\phi) \geqslant -C \right\} \tag{2.33}$$

is compact (for the L^1-topology) for each $C > 0$.

Definition 2.5. The *strong topology* of $\mathcal{E}^1$ is the coarsest refinement of the weak topology in which $E : \mathcal{E}^1 \to \mathbb{R}$ is continuous. In other words, $\phi_j \in \mathcal{E}^1$ converges to ϕ in the strong topology if and only if $\phi_j \to \phi$ in the weak topology (i.e. $\int_M |\phi_j - \phi|\omega^n \to 0$) and $E(\phi_j) \to E(\phi)$, denoted by $\phi_j \xrightarrow{s} \phi$.

Let $\omega_1, \cdots, \omega_n$ be Kähler forms. For any $\phi_j \in \mathrm{Psh}(\omega_j)$, $j = 1, \cdots, n$, we consider the canonical approximations

$$\phi_j^{(k)} := \max(\phi_j, -k) \in \mathrm{Psh}(\omega_j) \cap L^\infty(X). \tag{2.34}$$

Then the fundamental work of Bedford-Taylar [57] allows us to define

$$(\omega_1 + \mathrm{dd^c}\phi_1^{(k)}) \wedge \cdots \wedge (\omega_n + \mathrm{dd^c}\phi_n^{(k)}) \tag{2.35}$$

on X as a closed positive (n,n)-current, i.e. a measure. Indeed, we locally write $\omega_j = \mathrm{dd^c}\psi_j$ for some smooth functions ψ_j on $U \subset X$, so that $\omega_j + \mathrm{dd^c}\phi_j^{(k)} = \mathrm{dd^c}u_j^{(k)}$ on U with $u_j^{(k)} := \psi_j + \phi_j^{(k)}$. Then $u_j^{(k)}$ is a bounded psh function. Thus $u_1^{(k)}(\mathrm{dd^c}u_2^{(k)})$ is a well-defined current. We define

$$\mathrm{dd^c}u_1^{(k)} \wedge \mathrm{dd^c}u_2^{(k)} := \mathrm{dd^c}(u_1^{(k)}(\mathrm{dd^c}u_2^{(k)})). \tag{2.36}$$

It is easy to check that $\mathrm{dd^c}u_1^{(k)} \wedge \mathrm{dd^c}u_2^{(k)}$ is a closed positive $(2,2)$-current. Inductively, we can define

$$\mathrm{dd^c}u_1^{(k)} \wedge \cdots \wedge \mathrm{dd^c}u_n^{(k)} := \mathrm{dd^c}(u_1^{(k)}(\mathrm{dd^c}u_2^{(k)} \wedge \cdots \wedge \mathrm{dd^c}u_n^{(k)})). \tag{2.37}$$

In fact, this gives a global measure on X.

The heart of Bedford-Taylor's theory is the *maximal principle* (see [57]), which enables that the measures

$$\mu_k(\phi_1, \cdots, \phi_n) := \mathbf{1}_{\cap_{j=1}^n \{\phi_j > -k\}}(\omega_1 + \mathrm{dd^c}\phi_1^{(k)}) \wedge \cdots \wedge (\omega_n + \mathrm{dd^c}\phi_n^{(k)}) \tag{2.38}$$

form an increasing sequence of Borel measures, whose mass is uniformly bounded from above by $[\omega_1] \cdots [\omega_n]$.

Thus, we can define the *mixed Monge-Ampère measure* as

$$(\omega_1 + \mathrm{dd^c}\phi_1) \wedge \cdots \wedge (\omega_n + \mathrm{dd^c}\phi_n) := \lim_{k\to\infty} \mu_k(\phi_1, \cdots, \phi_n), \tag{2.39}$$

which is a positive Radon measure with total measure $\leqslant [\omega_1] \cdots [\omega_n]$ and symmetric, multilinear with respect to each ϕ.

For any $\phi \in \mathrm{Psh}$, its Monge-Ampère measure is defined as

$$\omega_{\phi}^{n} := (\omega + \mathrm{dd^c}\phi) \wedge \cdots \wedge (\omega + \mathrm{dd^c}\phi). \tag{2.40}$$

We state a proposition which gives a glimpse for the strong convergence in $\mathcal{E}^1$, see [58]Theorem 3.46.

Proposition 2.1. For $\phi_k, \phi \in \mathcal{E}^1$, if $\phi_k \xrightarrow{s} \phi$, then $\omega_{\phi_k}^n \to \omega_{\phi}^n$ weakly, and $\int_M |\phi_k - \phi|\omega_v^n \to 0$ for any $v \in \mathcal{E}^1$.

2.2.2 The structure of the space of finite energy potentials

2.2.2.1 Geodesic and energy functional

Mabuchi [59] proved that $\mathcal{H}$ is a Riemannian symmetric space of the constant negative curvature for the L^2-structure. Hence, we can compute its Riemannian connection and geodesic ray. Let $\phi_t \in \mathcal{H}$ be a smooth curve for an interval $I \subset \mathbb{R}$. By a simple computation, it is the *geodesic* if and only if it satisfies the following equation

$$\ddot{\phi}_t - |\partial\dot{\phi}_t|_t^2 = \ddot{\phi}_t - g_t^{i\bar{j}}\partial_i\dot{\phi}_t\partial_{\bar{j}}\dot{\phi}_t = 0, \tag{2.41}$$

where g_t is the metric of $\omega + dd^c\phi_t$. Consider

$$\mathbb{D}_I := \{\tau \in \mathbb{C}^* | -\log|\tau| \in I\}. \tag{2.42}$$

We can view ϕ_t as a function on $X \times \mathbb{D}_I$ by

$$\Phi(\cdot, \tau) := \phi_t(\cdot) \tag{2.43}$$

for $t := -\log|\tau|$ (in this thesis, we always use τ as coordinate of $\mathbb{D}_I$ or $\mathbb{C}$ and $|\tau| = e^{-t}$) and also denote by $\Phi : I \to \mathcal{H}$. When $I = (0, 1)$, we denote $\mathbb{D} = \mathbb{D}_I$.

Observed by Semmes [60] and Donaldson [61], ϕ_t is geodesic if and only if Φ satisfies the following homogeneous Monge-Ampère equation

$$(p_1^*\omega + \mathrm{dd^c}\Phi)^{n+1} = 0, \tag{2.44}$$

where $p_1 : X \times \mathbb{C} \to X$. From this point-view, we can extend the geodesic to $\mathcal{E}^1$.

Definition 2.6. (i) A *psh path* is a map $\Phi : I \to \mathrm{Psh}$ if the corresponding function on $X \times \mathbb{D}_I$ is $p_1^*\omega$-psh.

(ii) A psh path $\Psi : (0, 1) \to \mathrm{Psh}$ is *dominated* by ω_0-psh function ψ_0, ψ_1 if

$$\lim_{t\to 0}\phi_t \leqslant \psi_0, \quad \lim_{t\to 1}\phi_t \leqslant \psi_1. \tag{2.45}$$

If such Ψ exists, a simple envelope argument shows that there exists a largest one

$\Phi : I \to \mathrm{Psh}$, called *psh geodesic segment* joining ϕ_0 to ϕ_1,

$$\Phi = \sup_{\Psi \in \mathcal{S}} \Psi, \tag{2.46}$$

where

$$\mathcal{S} := \{\text{psh path } \Psi | \Psi \text{ is dominated by } \phi_0, \phi_1\}. \tag{2.47}$$

Lemma 2.2 (Darvas's survey [58][Lemma 3.14]). When $\phi_0, \phi_1 \in \mathrm{Psh} \cap L^\infty$, then such Φ given by (2.46) is the unique bounded $p_1^*\omega$-psh solution of the equation

$$\begin{cases} (p_1^*\omega + \mathrm{dd^c}\Phi)^{n+1} = 0, \\ \omega + \mathrm{dd^c}\Phi|_\tau \geqslant 0, \Phi \text{ is } \mathbb{S}^1 - \text{invariant}, \\ \lim_{t\to 0} \phi_t = \phi_0,\ \lim_{t\to 1} \phi_t = \phi_1. \end{cases} \tag{2.48}$$

For each psh path Φ, we have the following well-known computation (see [54][Proposition 6.2])

$$dd^c_\tau(E \circ \Phi) = (p_2)_* \left((p_1^*\omega + \mathrm{dd}^c_{(x,\tau)}\Phi)^{n+1} \right). \tag{2.49}$$

Hence, one obtains that E is convex along a psh path and is affine along a psh geodesic segment.

Definition 2.7. A map $\Phi : \mathbb{R}_{\geqslant 0} \to \mathcal{E}^1$ is called a *psh geodesic ray* if the restriction of Φ to each compact interval $[a, b]$ coincides (up to affine reparametrization) with the psh geodesic joining ϕ_a to ϕ_b.

We denote by $\mathcal{R}^1 = \mathcal{R}^1(X, \omega)$ the space of psh geodesic rays in $\mathcal{E}^1$ emanating from 0. We also write $\mathcal{R}^\infty = \mathcal{R}^\infty(X, \omega)$ for the set of locally bounded geodesic rays emanating from 0.

Mabuchi [62] defined a functional M on $\mathcal{H}$ whose critical points are cscK metrics, called *K-energy* or *Mabuchi functional*, given by its variation along a path $\phi_t = \phi + t\psi \in \mathcal{H}$ with $\psi \in C^\infty(X)$,

$$\frac{d}{dt}\Big|_{t=0} M(\phi_t) = \int_M \psi(\hat{S} - S_\phi)\omega_\phi^n. \tag{2.50}$$

He also computed the 2nd derivative of M along smooth geodesics and obtained that M is convex along smooth geodesics. This is an important property for M as a functional.

Unfortunately, in general, there is no the smooth geodesics in $\mathcal{H}$. Chen [63] showed that for any two elements in $\mathcal{H}$ can be joined by $C^{1,1}$-geodesic. Lempert and Vivas [64] proved that there does not always exist a C^3 geodesic between two smooth Kähler potentials. Darvas and Lempert [65] observed that the $C^{1,1}$-regularity is optimal.

Chen [44] and Tian [66] represented explicitly K-energy as

$$M(\phi) = H(\phi) + \overline{S}E(\phi) - nE_{\mathrm{Ric}(\omega_0)}(\phi), \tag{2.51}$$

where

$$H(\phi) = V^{-1}\int_X \log\left(\frac{\omega_\phi^n}{\omega^n}\right)\omega_\phi^n \tag{2.52}$$

is the entropy of the measure ω_ϕ^n with respect to ω^n, and E_χ is twisted Monge-Ampère energy, defined by

$$E_\chi(\phi) = \frac{1}{nV}\sum_{j=0}^{n-1}\int_X \phi\chi\wedge\omega_\phi^j\wedge\omega_\psi^{n-1-j}, \tag{2.53}$$

for a closed $(1,1)$-form χ. This is the so-called *Chen-Tian formula*.

Since the mixed Monge-Ampère measure is well-defined on $\mathcal{E}^1$, then one can extend the functional H and E_χ to $\mathcal{E}^1$. Thus M is a well-defined functional on $\mathcal{E}^1$ from this formula (2.51), due to Berman-Darvas-Lu [67].

The convexity of K-energy along $C^{1,1}$-geodesic or more general psh geodesic is a very crucial question to understand the structure of this functional and find its critical points. Berman-Berndtsson [50] show that K-energy is convex along a $C^{1,1}$-geodesic. Berman-Darvas-Lu [67] prove that K-energy is convex along a psh geodesic in $\mathcal{E}^1$.

2.2.2.2 Darvas's d_1-distance

By Darvas's works [68-69], he constructed a natural L^1-Finsler metric d_1 on $\mathcal{H}$ with the property $\mathcal{E}^1 = \overline{\mathcal{H}}^{d_1}$, defined as follows,

$$d_1(\phi,\psi) := \inf\left\{\int_0^1 \|\dot{\phi}_t\|_{L^1(\omega_{\phi_t})}dt \;\middle|\; (\phi_t)_{t\in[0,1]} \text{ is smooth path joining } \phi \text{ to } \psi\right\}, \tag{2.54}$$

for any $\phi,\psi\in\mathcal{H}$. If $\phi,\psi\in\mathcal{E}^1$, then there exist $\phi_k,\psi_k\in\mathcal{H}$ such that $\phi_k\searrow\phi$ and $\psi_k\searrow\psi$,

$$d_1(\phi,\psi) := \lim_{k\to\infty} d_1(\phi_k,\psi_k). \tag{2.55}$$

The d_1-distance is independent on the choice of sequences $\{\phi_k\}$ and $\{\psi_k\}$. He showed that $(\mathcal{E}^1, d_1)$ is a complete metric space. Moreover, metric topology induced by d_1 is nothing but strong topology. Any psh geodesic $\Phi : I \to \mathcal{E}^1$ in the sense of Definition 2.6

((ii)) is a constant speed geodesic for d_1, i.e., there is a nonnegative constant C, such that

$$d_1(\phi_t, \phi_s) = C|t - s|, \ \forall s, t \in I. \tag{2.56}$$

We refer to Darvas's survey [58] to the detail of this section.

Chapter 3 Volume of big divisors

Let X be a normal projective variety.

3.1 Volume function

In this subsection, we review some of the standard facts on the volume function.

Given a Cartier divisor D and a curve C on X, the *intersection number* is defined as

$$D \cdot C := \deg(\mathcal{O}_C(D|_C)). \tag{3.1}$$

By the Riemann-Roch theorem, this is well-defined. When D is a hypersurface that does not contain C, then the intersection number counts (with multiplicities) the number of points of intersection of D and C.

This formula implies that the intersection number is linear in D. Thus we may extend the definition of the intersection number $D \cdot C$ by linearity for a $\mathbb{R}$-Cartier divisor D and a curve C. Indeed, an $\mathbb{R}$-Cartier divisor D is represented by a finite sum $D = \sum c_i A_i$ where $c_i \in \mathbb{R}$ and A_i is a Cartier divisor. Then the intersection number is defined as

$$D \cdot C = \sum c_i (A_i \cdot C) \tag{3.2}$$

for any curve $C \subset X$.

We say that two $\mathbb{R}$-Cartier divisors D and D' are *numerically equivalent* if

$$D \cdot C = D' \cdot C \tag{3.3}$$

for any curve C on X, denoted by $D \equiv D'$.

The Néron-Severi space $N^1(X)$ is the real vector space of numerical equivalence classes of $\mathbb{R}$-Cartier divisors on X. In general, the Néron-Severi space is denoted by $N^1(X)_{\mathbb{R}}$. But for simplicity, we denote it by $N^1(X)$. For any Cartier divisor D, the *volume* of D is defined to be

$$\mathrm{Vol}(D) = \limsup_{m \to \infty} \frac{h^0(X, mD)}{m^n/n!}. \tag{3.4}$$

For any natural number $a > 0$, we have

$$\mathrm{Vol}(aD) = a^n \mathrm{Vol}(D). \tag{3.5}$$

It follows that the volume for any $\mathbb{Q}$-Cartier divisor D is defined to be

$$\mathrm{Vol}(D) = \frac{1}{a^n}\mathrm{Vol}(aD), \tag{3.6}$$

for some $a \in \mathbb{N}$, such that aD is Cartier divisor. This is independent of the choice of a. The volume of a $\mathbb{Q}$-Cartier divisor depends only on its numerical equivalence class. Thus, the volume function can be descended to $N^1(X)_{\mathbb{Q}}$. Then the volume function extends continuously to $N^1(X)$. The volume function satisfies the homogeneous property, i.e.

$$\mathrm{Vol}(aD) = a^n\mathrm{Vol}(D). \tag{3.7}$$

for any $a > 0$ and any D in $N^1(X)$.

We recall some definitions of positivity of $\mathbb{R}$-divisors. An $\mathbb{R}$-divisor D in $N^1(X)$ is called *nef* if the intersection number $L \cdot C$ is nonnegative for any curve C on X. The volume of a nef $\mathbb{R}$-divisor D is equal to the top self-intersection number D^n. All nef classes in $N^1(X)$ form a convex cone, called the *nef cone*, denoted by Nef(X), whose interior is called the *ample cone*, denoted by Amp(X). An $\mathbb{R}$-divisor D in $N^1(X)$ is called *big* if

$$\mathrm{Vol}(D) > 0. \tag{3.8}$$

All big classes in $N^1(X)$ form a convex open cone, called the *big cone*, denoted by Big(X), whose closure is called the *pseudo-effective* (*psef for short*) *cone*. For any two big $\mathbb{R}$-divisors D and B, one can obtain

$$\mathrm{Vol}(D + B) \geqslant \mathrm{Vol}(D). \tag{3.9}$$

For more details of the volume function, we refer to the standard reference [70].

3.2 Positive intersection product

In this subsection, we present some preliminaries about the positive intersection product. We follow the notions of [71]. References to this subsection are [72], [71], [73].

In general, the volume of a big divisor is not equal to its top self-intersection number. But it can be computed as the movable intersection number (see [70][Chapter 11]) by Fujita's approximation theorem. In other words, for any big divisor D, let $\pi_m : X_m \to X$ be the resolution of base locus $\mathfrak{b}(|mD|)$ with the exceptional divisor E_m and set $D_m = \pi_m^* D - \frac{1}{m}E_m$, then

$$\mathrm{Vol}(D) = \lim_m D_m^n. \tag{3.10}$$

To compute the volume of a big divisor, in [71] the authors introduced a valid notion, the so-called *positive intersection product*. Next we recall the notion (also see [71], [72] for details).

Recall that the Riemann-Zariski space of X is the locally ringed space defined by

$$\mathfrak{X} := \varprojlim_{\pi} X_{\pi}, \tag{3.11}$$

where X_π runs over all birational models of X with the birational morphism $\pi : X_\pi \to X$. Here the projective limit is taken in the category of locally ringed spaces. We do not use the theory of Riemann-Zariski spaces in an essential way in this paper.

For any smooth projective variety V of dimension n and any integer $0 \leqslant p \leqslant n$, let $N^p(V)$ be the real vector space of numerical equivalence classes of codimension p-cycles (see [74][Chapter 19]). Any birational morphism $\nu : V' \to V$ induces a pull-back morphism

$$\nu^* : N^p(V) \to N^p(V') \tag{3.12}$$

and a push-forward morphism

$$\nu_* : N^p(V') \to N^p(V). \tag{3.13}$$

There exists an intersection pairing $N^p(V) \times N^{n-p}(V) \to \mathbb{R}$, which is preserved under pull-back by birational morphisms, and for which push-forward and pull-back are adjoint to each other.

Definition 3.1 ([71][Definition 1.1]). For any integer $0 \leqslant p \leqslant n$,

- the space of p-codimensional Weil classes on the Riemann-Zariski space $\mathfrak{X}$ is defined as

$$N^p(\mathfrak{X}) := \varprojlim_{\pi} N^p(X_\pi), \tag{3.14}$$

  with arrows defined by push-forward,
- the space of p-codimensional Cartier classes on $\mathfrak{X}$ is defined as

$$CN^p(\mathfrak{X}) := \varinjlim_{\pi} N^p(X_\pi), \tag{3.15}$$

  with arrows defined by pullback.

By definition, a Weil class α in $N^p(\mathfrak{X})$ is given by its *incarnations* α_π in $N^p(X_\pi)$ on each smooth birational model of X, satisfying

$$\nu_*(\alpha_{\pi'}) = \alpha_{\pi''} \tag{3.16}$$

for any birational morphism $\nu : X_{\pi'} \to X_{\pi''}$ with $\pi' = \pi'' \circ \nu$.

On the other hand, since $\nu_*\nu^*\alpha = \alpha$ for any birational morphism $\nu : X_{\pi'} \to X_{\pi''}$ and any $\alpha \in N^p(X_{\pi''})$, then it induces an injection

$$CN^p(\mathfrak{X}) \hookrightarrow N^p(\mathfrak{X}), \tag{3.17}$$

i.e., a Cartier class is a Weil class. Concretely, a Weil class α is Cartier iff there exists π such that its incarnation $\alpha_{\pi'}$ on the higher blow-ups $X_{\pi'}$ are obtained by pulling back α_π.

Further for each π, given a class α in $N^p(X_\pi)$, one can extend it to a Cartier class by pullback of it. Thus, we have the natural injection

$$N^p(X_\pi) \hookrightarrow CN^p(\mathfrak{X}), \tag{3.18}$$

When $p = 1$, we refer to the space $CN^1(\mathfrak{X})$ as the *Néron-Severi space* of $\mathfrak{X}$. Its elements are the so-called *Shokurov's b-divisors*.

In the sequel, we use the notation $\alpha \geqslant 0$ for a psef class α in $N^p(X)$ (see [74]). We consider positive Cartier classes in $\mathfrak{X}$. For a birational morphism $\nu : V' \to V$, a class α in $N^1(V)$ is nef (resp. psef, big) if and only if $\nu^*\alpha$ is nef (resp. psef, big). Therefore, one can extend these definitions to the Riemann-Zariski space.

Definition 3.2 ([71][Definition 1.6]). A Cartier class $\alpha \in CN^1(\mathfrak{X})$ is called nef (resp. psef, big) if its incarnation α_π is nef (resp. psef, big) for some π.

On a smooth projective variety V, for any classes $\alpha_1, \cdots, \alpha_p \in N^1(V)$, the intersection product $\alpha_1 \cdots \alpha_p$ belongs to $N^p(V)$ (see [74]). Further for any birational morphism $\nu : V' \to V$, one has $\nu^*\alpha_1 \cdots \nu^*\alpha_p = \nu^*(\alpha_1 \cdots \alpha_p)$, see [74][Chapter 19]. One can define the intersection product of p-Cartier classes $\alpha_1, \cdots, \alpha_p \in CN^1(\mathfrak{X})$, which have a common determination X_π, as the Cartier class in $CN^p(\mathfrak{X})$ determined by $\alpha_{1,\pi} \cdots \alpha_{p,\pi}$.

Definition 3.3 ([71][Definition 2.5]). For any big classes $\alpha_1, \cdots, \alpha_p \in CN^1(\mathfrak{X})$, their *positive intersection product*

$$\langle \alpha_1 \cdots \alpha_p \rangle \in N^p(\mathfrak{X}) \tag{3.19}$$

is defined as the least upper bound of the set of classes

$$(\alpha_1 - D_1) \cdots (\alpha_p - D_p) \in N^p(\mathfrak{X}) \tag{3.20}$$

where D_i is an effective Cartier $\mathbb{Q}$-divisor on $\mathfrak{X}$ such that $\alpha_i - D_i$ is nef.

Remark 3.1. In [72][Theorem 3.5], the authors give an analytic definition of the positive intersection product (they call it as the movable intersection product) for Kähler manifolds. For any *big classes* $\alpha_1, \cdots, \alpha_p$ on the Kähler manifold V, in which the big class means that each α_j can be represented by a *Kähler current* T, i.e. a closed positive $(1, 1)$-current T

such that $T \geqslant \theta\omega$ for some smooth Hermitian metric ω and a small constant $\theta > 0$, one defines

$$\langle \alpha_1 \cdots \alpha_p \rangle := \sup_{T_j, \pi} \pi_*(\gamma_1 \wedge \cdots \wedge \gamma_p) \tag{3.21}$$

where $T_j \in \alpha_j$ is a Kähler current with logarithmic poles, i.e. there is a modification $\pi_j : V_j' \to V$ such that $\pi_j^* T_j = [E_j] + \gamma_j$ for some effective $\mathbb{Q}$-divisor E_j and closed semi-positive form γ_j. Here we take a common resolution $\pi : V' \to V$, and write

$$\pi^* T_j = [E_j] + \gamma_j. \tag{3.22}$$

Definition 3.4 ([71][Definition 2.10]). For any psef classes $\alpha_1, \cdots, \alpha_p \in CN^1(\mathfrak{X})$, their positive intersection product

$$\langle \alpha_1 \cdots \alpha_p \rangle \in N^p(\mathfrak{X}) \tag{3.23}$$

is defined as the limit

$$\lim_{\varepsilon \to 0^+} \langle (\alpha_1 + \varepsilon\gamma) \cdots (\alpha_p + \varepsilon\gamma) \rangle, \tag{3.24}$$

where γ in $CN^1(\mathfrak{X})$ is any big Cartier class.

This definition is independent of the choice of the big class γ (see [71][Definition 2.10]). In fact, if $\alpha_1, \cdots, \alpha_p \in CN^1(\mathfrak{X})$ are nef classes, then

$$\langle \alpha_1 \cdots \alpha_p \rangle = \alpha_1 \cdots \alpha_p. \tag{3.25}$$

For any big $\mathbb{R}$-divisor $D \in N^1(X)$, we have

$$\mathrm{Vol}(D) = \langle D^n \rangle, \tag{3.26}$$

see [71][Theorem 3.1], also see [75][Definition 1.17] for an definition by pluripotential theory.

An analytic definition of the volume of the big classes is given in [72][Definition 3.2] (see Remark 3.1). Concretely, the volume of a big class α is define as

$$\mathrm{Vol}(\alpha) := \sup_{T \in \alpha} \int \beta^n > 0, \tag{3.27}$$

where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles, and $\mu^* T = [E] + \beta$ with respect to some resolution $\nu : \widetilde{X} \to X$ such that E is an effective $\mathbb{Q}$-divisor and β is a closed semi-positive form on $\widetilde{X}$.

An interesting fact about the volume function on the big cone, due to Boucksom-Favre-Jonsson [71], is stated as follows,

Theorem 3.1 ([71][Theorem A]). The volume function is C^1-differentiable on the big cone of $N^1(X)$. If $\alpha \in N^1(X)$ is big and $\gamma \in N^1(X)$ is arbitrary, then

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(\alpha + t\gamma) = n\langle \alpha^{n-1} \rangle \cdot \gamma. \tag{3.28}$$

We collect some facts about the positive intersection product as follows, for using later,

Proposition 3.1 ([71]Proposition 2.9, Corollary 3.6). (i) The positive intersection product is symmetric, homogeneous of degree 1, and super-additive in each variable. Moreover, it is continuous on the *p*-fold product of the big cone of $CN^1(\mathfrak{X})$.

(ii) For any psef class α in $CN^1(\mathfrak{X})$, one obtain

$$\langle \alpha^n \rangle = \langle \alpha^{n-1} \rangle \cdot \alpha. \tag{3.29}$$

Remark 3.2. In general, the positive intersection product is not multilinear, see [75]Definition 1.17 for an analytic explanation.

Chapter 4 K-stability and non-Archimedean formulation

In this Chapter, we introduce K-stability from the original algebro-geometric definition of Donaldson [10] and the non-Archimedean formulation [19]. From now on, we assume that X is an n-dimensional normal projective $\mathbb{Q}$-Gorenstein variety, which means that the canonical divisor of X is $\mathbb{Q}$-Cartier.

Let (X, L) be a polarized variety, namely, X is a projective variety and L is an ample line bundle on X.

4.1 Test configuration and K-stability

Now, we state the definition of test configuration and K-stability from [15, 19].

Definition 4.1. A *test configuration* $(\mathcal{X}, \mathcal{L})$ for (X, L) consists of the following data:

(i) a flat and proper morphism of schemes $\pi : \mathcal{X} \to \mathbb{C}$;

(ii) a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{C}$;

(iii) a $\mathbb{C}^*$-linearized $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$;

(iv) isomorphism $(\mathcal{X}_t, \mathcal{L}_t) := (\pi^{-1}(t), \mathcal{L}|_{\pi^{-1}(t)}) \cong (X, L)$ for any $t \neq 0$.

A test configuration $(\mathcal{X}, \mathcal{L})$ is called *normal* (resp. *ample*, resp. *semiample*) if $\mathcal{X}$ is normal (resp. $\mathcal{L}$ is π-relative ample, resp. π-relative semiample).

Example 4.1. A *product* test configuration is $(X, L) \times \mathbb{C}$ with a diagonal $\mathbb{C}^*$-action. A *trivial* test configuration is a product test configuration with diagonal $\mathbb{C}^*$-action trivially acting on X, denoted by $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$.

If there is a $\mathbb{C}^*$-equivariant birational morphism $\mathcal{X}_1 \to \mathcal{X}_2$ between any two given test configurations, then we say $\mathcal{X}_1$ *dominates* $\mathcal{X}_2$. A test configuration $\mathcal{X}$ is called *dominating* if $\mathcal{X}$ dominates $X_{\mathbb{A}^1}$. Any two test configurations can be dominated by a third.

We say that two test configurations $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ for (X, L) are *equivalent* if the pullbacks of $\mathcal{L}$ and $\mathcal{L}'$ to some test configuration $\mathcal{X}''$ dominating $\mathcal{X}$ and $\mathcal{X}'$ coincide. We will see in Section 4.2 why we define the equivalence of test configurations.

For each $m \in \mathbb{N}$, we have a vector space

$$H_m := H^0(\mathcal{X}_0, m\mathcal{L}_0) \tag{4.1}$$

with a $\mathbb{C}^*$-action. We denote by $d_m := \dim H_m$ and w_m the weight of the induced action on $\wedge^{d_m} H_m$, which is a polynomial of degree at most $n + 1$ by the equivariant Riemann-

Roch theorem (or see [19][Theorem 3.1]). Then we have an asymptotic expansion

$$\frac{w_m}{md_m} := F_0 + F_1 m^{-1} + F_2 m^{-2} + \cdots . \tag{4.2}$$

The *Donaldson-Futaki invariant* of $(\mathcal{X}, \mathcal{L})$ is defined as

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) := -2F_1. \tag{4.3}$$

When the central fiber $\mathcal{X}_0$ is smooth, as computed by Donaldson in [10], the Donalson-Futaki invariant is the original Futaki invariant (2.20).

Definition 4.2. The polarized variety (X, L) is

- *K-semistable* if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geqslant 0$ for all semiample test configurations $(\mathcal{X}, \mathcal{L})$;
- *K-stable* if it is K-semistable, and has $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = 0$ only when $(\mathcal{X}, \mathcal{L})$ is trivial test configuration;
- *K-polystable* if it is K-semistable, and has $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = 0$ only when $(\mathcal{X}, \mathcal{L})$ is product test configuration;
- *uniform K-stable* if for all semiample test configurations, there exists a uniform constant $\delta > 0$ satisfies

  $$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geqslant \delta \|(\mathcal{X}, \mathcal{L})\|, \tag{4.4}$$

  where $\|(\mathcal{X}, \mathcal{L})\|$ represents some norm of test configuration $(\mathcal{X}, \mathcal{L})$, defined later, which is just non-Archimedean J-functional J^{NA}.

Later, Odaka [76] and Wang [77] gave an intersection formula of Donaldson-Futaki invariant,

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{(K_{\overline{\mathcal{X}}/\mathbb{P}^1} \cdot \overline{\mathcal{L}}^n)}{\mathrm{Vol}(L)} + \hat{S} \frac{(\overline{\mathcal{L}}^{n+1})}{(n+1)\mathrm{Vol}(L)}, \tag{4.5}$$

where $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ is the compactification of $(\mathcal{X}, \mathcal{L})$ over $\mathbb{P}^1$ by adding (X, L) at $\infty \in \mathbb{P}^1$. This formula looks like more convenient than the original definition (4.3) to compute.

4.2 Non-Archimedean metrics and functionals

In this section, we introduce the non-Archimedean formulation for K-stability, and focus on representing test configurations as non-Archimedean metrics. We refer to [19], [78].

In this thesis, a *valuation* on X means a real-valued valuation $v : \mathbb{C}(X)^* \to \mathbb{R}$, trivial on $\mathbb{C}$. We denote by X^{val} the space of valuations. The *center* $c(v) := c_X(v) \in X$ is characterized as the unique (scheme) point $\xi \in X$ such that $v \geqslant 0$ on the local ring $\mathcal{O}_{X,\xi}$

and $v > 0$ on its maximal ideal.

Denoted by X^{an} the *Berkovich analytification* of X with respect to the trivial absolute value on $\mathbb{C}$. We view X^{an} as a compact topological space, whose points can be understood as semivaluations on X, i.e. valuation $v : \mathbb{C}(Y)^* \to \mathbb{R}$ on the function field of a subvariety Y of X, trivial on $\mathbb{C}$ (Here, we do not give the explicit definition of Berkovich analytification, see [79]). The subvariety Y is called the *support* of v, denoted by $\mathrm{Supp}(v)$. As a set, we have $X^{\mathrm{an}} = \coprod_Y Y^{val}$ with Y running over all irreducible subvarieties of X, and the topology of X^{an} is the coarsest topology such that for each Zariski open set $U \subset X$, we have:

- the set $U^{\mathrm{an}} := \{v \in X^{\mathrm{an}} \mid \mathrm{Supp}(v) \cap U \neq \emptyset\}$ is open;
- for each $f \in \mathcal{O}(U)$, the function $|f| : U^{\mathrm{an}} \to \mathbb{R}_+$ defined by $|f|(v) := e^{-v(f)}$ is continuous.

There exists a continuous map $\ker : X^{\mathrm{an}} \to X$ sending v to the generic point $\ker(v)$ of its support, called the *kernal map*. The semivaluation v gives rise to a multiplicative norm $|\cdot|_v$ on the residue field $\mathbb{C}(\ker(v))$ as follows,

$$|f|_v := e^{-v(f)} \tag{4.6}$$

for any $f \in \mathbb{C}(\ker(v))$, which is a non-Archimedean norm, i.e. satisfying $|f+g|_v \leqslant \max\{|f|_v, |g|_v\}$ for any $f, g \in \mathbb{C}(\ker(v))$. We denote by $\mathcal{H}(v)$ the completion of $\mathbb{C}(\ker(v))$ with respect to the norm $|\cdot|_v$.

Given any ideal $\mathfrak{a} \subset \mathcal{O}_X$ and $v \in X^{\mathrm{val}}$ with the center $c(v) \in X$, one sets

$$v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a}_{c(v)}\}. \tag{4.7}$$

In particular, we can just consider the divisorial valuations. Denoted by $X^{\mathrm{div}}_{\mathbb{Q}}$ the set of rational divisorial valuations on X, i.e., $v \in X^{\mathrm{div}}_{\mathbb{Q}} : \mathbb{C}(X)^* \to \mathbb{Q}$, such that $v = c\,\mathrm{ord}_F$ for some $c \in \mathbb{Q}$ and prime divisor F over X, which means that there exists a normal birational model Y of X and F is a prime divisor on Y. The *log discrepancy* of $v \in X^{\mathrm{div}}_{\mathbb{Q}}$ is defined as

$$A_X(v) := c(1 + \mathrm{ord}_F(K_{Y/X})), \tag{4.8}$$

where $K_{Y/X}$ is the relative canonical divisor. The *trivial valuation* is defined as $v_{\mathrm{triv}}(f) := 0$ for any $f \in \mathbb{C}(X)^*$. For convenience, we set $A_X(v_{\mathrm{triv}}) = 0$.

The first factor projection p_1 induces a map $(X \times \mathbb{C})^{\mathrm{div}}_{\mathbb{Q}} \to X^{\mathrm{div}}_{\mathbb{Q}}$. This has a canonical section $\sigma : X^{\mathrm{div}}_{\mathbb{Q}} \to (X \times \mathbb{C})^{\mathrm{div}}_{\mathbb{Q}}$, called *Gauss extension*, defined by

$$\sigma(v)\left(\sum_i f_i \tau^i\right) := \min_i \{v(f_i) + i\}, \tag{4.9}$$

for each finite $f_0, \cdots, f_r \in \mathbb{C}(X)$. For each $v \in X^{\mathrm{div}}_{\mathbb{Q}}$, one has

$$A_{X\times\mathbb{C}}(\sigma(v)) = A_X(v) - 1, \tag{4.10}$$

see [19][section 4]. Furthermore, each test configuration $\mathcal{X}$ for X gives an embedding

$$\sigma_{\mathcal{X}} : X^{\mathrm{an}} \hookrightarrow (\mathcal{X}\backslash\mathcal{X}_0)^{\mathrm{an}} \subset \overline{\mathcal{X}}^{\mathrm{an}}. \tag{4.11}$$

A *vertical* $\mathbb{R}$*-Cartier divisor* on $\mathcal{X}$ means a $\mathbb{C}^*$-invariant $\mathbb{R}$-Cartier divisor with support in $\mathcal{X}_0$. Such divisors forms a finite dimensional $\mathbb{R}$-vector space, denoted by $\mathrm{VCar}(\mathcal{X})_{\mathbb{R}}$.

Each $D \in \mathrm{VCar}(\mathcal{X})_{\mathbb{Q}}$ defines a continuous function φ_D

$$\varphi_D(v) := \sigma_{\mathcal{X}}(D) \tag{4.12}$$

for $v \in X^{\mathrm{div}}$, where the right-hand side is defined as $m^{-1}\sigma_{\mathcal{X}}(v)(\mathcal{O}_{\mathcal{X}}(-mD))$ for any choice $m \in \mathbb{Z}_{>0}$ such that mD is a Cartier divisor. We do not explain the explicit definition of the continuity here, which is for Berkovich topology.

If $\rho : \mathcal{X}' \to \mathcal{X}$ is a morphism between test configurations, then $\sigma_{\mathcal{X}'}(v)(\rho^* D) = \sigma_{\mathcal{X}}(v)(D)$ for any $v \in X^{\mathrm{div}}$, and hence $\varphi_{\rho^* D} = \varphi_D$. It gives rise to a $\mathbb{Q}$-linear map

$$\mathrm{PL}(X^{\mathrm{an}}) := \varinjlim_{\mathcal{X}} \mathrm{VCar}(\mathcal{X})_{\mathbb{Q}} \to C^0(X^{\mathrm{an}}). \tag{4.13}$$

An important fact is the density of $\mathrm{PL}(X^{\mathrm{an}})$.

Theorem 4.1 ([78][Theorem 2.2]). The space $\mathrm{PL}(X^{\mathrm{an}})$ is dense in $C^0(X^{\mathrm{an}})$ with respect to the topology of uniform convergence.

Let Q be a $\mathbb{Q}$-line bundle on X. For each test configuration $(\mathcal{X}, \mathcal{Q})$ for (X, Q), we pick a common resolution $\mathcal{X}'$, with morphisms $\rho : \mathcal{X}' \to \mathcal{X}$ and $\pi : \mathcal{X}' \to X \times \mathbb{C}$. Then

$$D := \rho^* \mathcal{Q} - \pi^* p_1^* Q \in \mathrm{VCar}(\mathcal{X}')_{\mathbb{Q}}, \tag{4.14}$$

and mapping $\mathcal{Q}$ to D gives a one-to-one correspondence between the set of equivalence classes of test configurations for (X, Q) and the $\mathbb{Q}$-vector space $\varinjlim_{\mathcal{X}} \mathrm{VCar}(\mathcal{X})_{\mathbb{Q}}$. Then we define

$$\varphi_{(\mathcal{X},\mathcal{Q})} := \varphi_D. \tag{4.15}$$

A (*non-Archimedean*) *line bundle* L^{an} on X^{an} means the analytification of the total space of a line bundle L on X. By the GAGA result for Berkovich analytification in [79]Section 3.5, one has a canonical map $p^{\mathrm{an}} : L^{\mathrm{an}} \to X^{\mathrm{an}}$, which is the analytification of the corresponding map $L \to X$. The fiber $L^{\mathrm{an}}_v := (p^{\mathrm{an}})^{-1}(v)$ over a point $v \in X^{\mathrm{an}}$ is isomorphic to the Berkovich affine line over the complete residue field $\mathcal{H}(v)$, i.e. $(\mathrm{Spec}\mathcal{H}(v)[T])^{\mathrm{an}}$, where T is a formal variable.

Denote by $L^{\mathrm{an},\times}$ the complement in L^{an} of the zero section. A (*non-Archimedean*) *metric* on L^{an} is a function $\phi^{\mathrm{an}} : L^{\mathrm{an},\times} \to \mathbb{R}$ such that $|\cdot|_{\phi^{\mathrm{an}}} = e^{-\phi^{\mathrm{an}}} : L^{\mathrm{an},\times} \to \mathbb{R}^{\times}_{+}$ behaves like a norm on each fiber L^{an}_v (see [79]). Parallelling with the complex side, if ϕ^{an} is a metric on L^{an}, then any other metric is of the form $\phi^{\mathrm{an}} + \varphi$, where φ is a function on X^{an}.

Any line bundle L^{an} on X^{an} admits a *trivial metric* φ_{triv} as follows. Given a point $v \in X^{\mathrm{an}}$, let ξ be a nonvanishing section of L on an open neighborhood $U \subset X$ of ξ. Then ξ defines a nonvanishing analytic section of L^{an} on the Zariski open neighborhood U^{an} of v in X^{an}, and $\varphi_{\mathrm{triv}}(\xi(v)) = 0$. The trivial metric allows us to consider metrics on L^{an} as functions on X^{an}. We will always use this notation.

Definition 4.3. A non-Archimedean metric φ, defined by (4.15) is called *positive* if some representative $(\mathcal{X}, \mathcal{L})$ of φ is semiample.

Denoted by $\mathcal{H}^{\mathrm{NA}}(L)$ the set of non-Archimedean positive metrics on L^{an}.

In [19], the authors defined the non-Archimedean version of the usual functionals on $\mathcal{H}$ as functionals on $\mathcal{H}^{\mathrm{NA}}(L)$. For any $\varphi = \varphi_{(\mathcal{X},\mathcal{L})} \in \mathcal{H}^{\mathrm{NA}}(L)$, we have the following definition of non-Archimedean functionals by the intersection formula:

$$
\begin{aligned}
E^{\mathrm{NA}}(\varphi) &:= \frac{1}{n+1}(\overline{\mathcal{L}}^{n+1}) && (4.16)\\
I^{\mathrm{NA}}(\varphi) &:= (\overline{\mathcal{L}} \cdot L^{n}_{\mathbb{P}^1}) - (\overline{\mathcal{L}}^{n+1}) + (\overline{\mathcal{L}}^{n} \cdot L_{\mathbb{P}^1}) && (4.17)\\
\Lambda^{\mathrm{NA}}(\varphi) &:= (\overline{\mathcal{L}} \cdot L^{n}_{\mathbb{P}^1}) && (4.18)\\
J^{\mathrm{NA}}(\varphi) &:= \Lambda^{\mathrm{NA}}(\varphi) - E^{\mathrm{NA}}(\varphi) && (4.19)\\
(E^{Q})^{\mathrm{NA}}(\varphi) &:= (\rho^* p_1^* Q \cdot \overline{\mathcal{L}}^{n}) && (4.20)\\
R^{\mathrm{NA}}(\varphi) &:= (K^{\log}_{X_{\mathbb{P}^1}/\mathbb{P}^1} \cdot \overline{\mathcal{L}}^{n}) && (4.21)\\
H^{\mathrm{NA}}(\varphi) &:= (K^{\log}_{\overline{\mathcal{X}}/\mathbb{P}^1} \cdot \overline{\mathcal{L}}^{n}) - (K^{\log}_{X_{\mathbb{P}^1}/\mathbb{P}^1} \cdot \overline{\mathcal{L}}^{n}) && (4.22)\\
M^{\mathrm{NA}}(\varphi) &= H^{\mathrm{NA}}(\varphi) + \hat{S} E^{\mathrm{NA}}(\varphi) + R^{\mathrm{NA}}(\varphi)\\
&:= (K^{\log}_{\overline{\mathcal{X}}/\mathbb{P}^1} \cdot \overline{\mathcal{L}}^{n}) + \frac{\hat{S}}{n+1}(\overline{\mathcal{L}}^{n+1}), && (4.23)
\end{aligned}
$$

where Q is a line bundle on X, $\rho : \overline{\mathcal{X}} \to X_{\mathbb{P}^1}$ and $L_{\mathbb{P}^1} = p_1^* L$. Similar with the original Aubin I, J-functionals, functionals I^{NA} and J^{NA} are nonnegative on $\mathcal{H}^{\mathrm{NA}}(L)$, and have the following equivalent relation

$$\frac{1}{n}J^{\mathrm{NA}} \leqslant I^{\mathrm{NA}} - J^{\mathrm{NA}} \leqslant nJ^{\mathrm{NA}}. \tag{4.24}$$

By (4.5), we have

$$M^{\mathrm{NA}}(\varphi) = \mathrm{Fut}(\mathcal{X}, \mathcal{L}) + V^{-1}((\mathcal{X}_{0,\mathrm{red}} - \mathcal{X}_0) \cdot \overline{\mathcal{L}^n}), \tag{4.25}$$

where $\mathcal{X}_{0,\mathrm{red}} = \sum_E E$ is the reduced part of central fiber if we write $\mathcal{X}_0 = \sum_E b_E E$. It follows that if central fiber $\mathcal{X}_0$ is reduced, then

$$M^{\mathrm{NA}}(\varphi) = \mathrm{Fut}(\mathcal{X}, \mathcal{L}). \tag{4.26}$$

In fact, by algebro-geometric theory, reducedness is not a difficulty. We can do a base change to get a reduced test configuration $(\mathcal{X}_d, \mathcal{L}_d) =: \varphi_d$ from any non-reduced test configuration $(\mathcal{X}, \mathcal{L})$ for d sufficiently large (see [19]$^{\text{Proposition 7.16}}$). Then one has

$$\mathrm{Fut}(\mathcal{X}_d, \mathcal{L}_d) = M^{\mathrm{NA}}(\varphi_d) = dM^{\mathrm{NA}}(\varphi). \tag{4.27}$$

Hence, K-stability conditions can be stated in terms of M^{NA}. In particular, uniform K-stability is equivalent to the following coercivity of M^{NA}

$$M^{\mathrm{NA}} \geqslant \delta J^{\mathrm{NA}} \ \text{ on } \mathcal{H}^{\mathrm{NA}}, \tag{4.28}$$

for some positive constant δ.

In both sides of Archimedean and non-Archimedean setting, they have similar story. The connection of them is that non-Archimedean version F^{NA} of a functional F should compute the slopes at infinity of F along psh rays with algebraic singularities, stated in later.

Remark 4.1. Here, we use the definition of uniform K-stability in [19]. One also can see [20]. The difference of these two formulation is norm functional. In [20], he used the so-called *minimum norm* $\|(\mathcal{X}, \mathcal{L})\|_m$. In fact,

$$V^{-1}\|(\mathcal{X}, \mathcal{L})\|_m = I^{\mathrm{NA}}(\varphi) - J^{\mathrm{NA}}(\varphi). \tag{4.29}$$

So these two definitions are same.

4.3 Filtrations and test configurations

A useful tool to study test configurations is the filtration, which is studied in [80], [81], [19] and so on. In this section, we recall the basic theory of filtrations.

We denote the section ring of L by

$$R = R(X, L) := \bigoplus_{m=0}^{\infty} R_m = \bigoplus_{m=0}^{\infty} H^0(X, mL), \tag{4.30}$$

which is a graded $\mathbb{C}$-algebra.

Definition 4.4. A (*graded*) *filtration* $\mathscr{F}$ of the section ring R consists of a family of subspace $\{\mathscr{F}^\lambda R_m\}_\lambda$of R_m, for $\lambda \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, satisfying:

(i) (decreasing) $\mathscr{F}^\lambda R_m \subset \mathscr{F}^{\lambda'} R_m$ if $\lambda \geqslant \lambda'$;

(ii) (left-continuous) $\mathscr{F}^\lambda R_m = \cap_{\lambda'<\lambda} \mathscr{F}^{\lambda'} R_m$;

(iii) (multiplicative) $\mathscr{F}^\lambda R_m \cdot \mathscr{F}^{\lambda'} R_{m'} \subset \mathscr{F}^{\lambda+\lambda'} R_{m+m'}$ for any $\lambda, \lambda' \in \mathbb{R}$ and $m, m' \in \mathbb{Z}_{\geqslant 0}$;

(iv) (linearly bounded) There exists $e_-, e_+ \in \mathbb{Z}$ such that $\mathscr{F}^{me_-} R_m = R_m$ and $\mathscr{F}^{me_+} R_m = 0$ for all $m \in \mathbb{Z}_{\geqslant 0}$.

A filtration $\mathscr{F}$ is called a $\mathbb{Z}$-*filtration* if $\mathscr{F}^\lambda = \mathscr{F}^{\lfloor \lambda \rfloor}$ for all $\lambda \in \mathbb{Z}$, which means that all jumping numbers of $\mathscr{F}$ are integers.

A $\mathbb{Z}$-filtration $\mathscr{F}$ is called *finitely generated* if the bigraded algebra

$$\bigoplus_{\lambda \in \mathbb{Z}, m \in \mathbb{Z}_{\geqslant 0}} \mathscr{F}^\lambda R_m \tag{4.31}$$

is finitely generated over $\mathbb{C}$.

For any filtration $\mathscr{F}$, we set

$$\lambda_{\min}^{(m)} := \inf\{\lambda \in \mathbb{R} \mid \mathscr{F}^\lambda R_m \neq R_m\}, \qquad \lambda_{\max}^{(m)} := \sup\{\lambda \in \mathbb{R} \mid \mathscr{F}^\lambda R_m \neq 0\}, \tag{4.32}$$

and

$$\lambda_{\min} := \lim_{m\to\infty} \frac{\lambda_{\min}^{(m)}}{m}, \qquad \lambda_{\max} := \lim_{m\to\infty} \frac{\lambda_{\max}^{(m)}}{m}. \tag{4.33}$$

For each $\lambda \in \mathbb{R}$, one defines a graded subalgebra of R by setting

$$R^{(\lambda)} := \bigoplus_{m \in \mathbb{N}} \mathscr{F}^{m\lambda} R_m. \tag{4.34}$$

The volume of the graded subalgebra $R^{(\lambda)}$ is defined as

$$\mathrm{Vol}(R^{(\lambda)}) := \lim_{m\to\infty} \frac{n!}{m^n} \dim \mathscr{F}^{m\lambda} R_m. \tag{4.35}$$

In [82], the authors associated a measure on $\mathbb{R}$ to a filtration.

Theorem 4.2 ([82]). Let $\mathscr{F}$ be a filtration of R and $\lambda_j^{(m)}$ be the jumping number of $\mathscr{F}^\lambda R_m$.

For each m, set

$$\nu_m := \frac{1}{d_m} \sum_j \delta_{m^{-1}\lambda_j^{(m)}} = -\frac{d}{d\lambda} \frac{\dim \mathscr{F}^{m\lambda} R_m}{d_m}, \tag{4.36}$$

which is a probability measure on $\mathbb{R}$. Then ν_m has uniformly bounded support and converges weakly as $m \to \infty$ to the probability measure

$$\nu := -\mathrm{Vol}(L)^{-1} \frac{d}{d\lambda} \mathrm{Vol}(R^{(\lambda)}), \tag{4.37}$$

called the *limit measure* of the filtration $\mathscr{F}$. Moreover, the support of ν is $[\lambda_{\min}, \lambda_{\max}]$.

In [81], the author constructed a $\mathbb{Z}$-filtration for each test configuration. Given a test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L), one defines a filtration $\mathscr{F}_{(\mathcal{X},\mathcal{L})}$ as

$$\mathscr{F}^{\lambda}_{(\mathcal{X},\mathcal{L})} R_m = \{ s \in H^0(X, mL) \mid \tau^{-\lceil \lambda \rceil} \bar{s} \in H^0(\mathcal{X}, m\mathcal{L}) \}, \tag{4.38}$$

where $\bar{s} \in H^0(\mathcal{X} \backslash \mathcal{X}_0, m\mathcal{L})$ denotes the $\mathbb{C}^*$-invariant section defined by $s \in H^0(X, mL)$ and τ is the coordinate of $\mathbb{C}$ as in Section 2.2.1. In fact, when $(\mathcal{X}, \mathcal{L})$ is an ample test configuration, then $\mathscr{F}_{(\mathcal{X},\mathcal{L})}$ is finitely generated.

Conversely, given a finitely generated filtration $\mathscr{F}$, then the Rees algebra of $\mathscr{F}$ defined as follows

$$\mathrm{Ree}(\mathscr{F}) := \bigoplus_{m \in \mathbb{N}} \left(\bigoplus_{\lambda \in \mathbb{Z}} \tau^{-\lambda} \mathscr{F}^{\lambda} R_m \right) \tag{4.39}$$

is a finitely generated $\mathbb{C}[\tau]$-algebra and generated in degree $m = 1$. We obtain an ample test configuration

$$\mathcal{X} := \mathrm{Proj}_{\mathbb{A}^1}(\mathrm{Ree}(\mathscr{F})), \quad \text{and} \quad \mathcal{L} := \mathcal{O}_{\mathcal{X}}(1). \tag{4.40}$$

Proposition 4.1 (see [19][proposition 2.15]). The above construction is a one-to-one correspondence between ample test configurations of (X, L) and finitely generated $\mathbb{Z}$-filtrations of the section ring R.

The *Duistermaat-Heckman measure* of a semiample test configuration $(\mathcal{X}, \mathcal{L})$ is the limit measure of the filtration $\mathscr{F}_{(\mathcal{X},\mathcal{L})}$.

In [19], the authors re-interpret the filtration $\mathscr{F}_{(\mathcal{X},\mathcal{L})}$ in terms of valuations.

Theorem 4.3 ([19][Theorem 5.16]). Let $(\mathcal{X}, \mathcal{L})$ be a normal, semiample test configuration of (X, L) dominating $X_{\mathbb{C}}$, and write $\mathcal{L} := \rho^* p_1^* L + D$ with the canonical morphism $\rho : \mathcal{X} \to X_{\mathbb{C}}$. Then we have

$$\mathscr{F}^{\lambda}_{(\mathcal{X},\mathcal{L})} R_m = \bigcap_E \{ s \in R_m \mid v_E(s) + m b_E^{-1} \mathrm{ord}_E(D) \geqslant \lambda \} \tag{4.41}$$

for all m divisible enough and all $\lambda \in \mathbb{Z}$, where E run over the irreducible components of

$\mathcal{X}_0$, $b_E := \mathrm{ord}_E(\mathcal{X}_0) = \mathrm{ord}_E(\tau)$, and $v_E := b_E^{-1} res(\mathrm{ord}_E)$ is the Rees valuation of $\mathcal{X}$.

Moreover, the support $[\lambda_{\min}, \lambda_{\max}]$ of the Duistermaat-Heckman measure of $\mathscr{F}_{(\mathcal{X},\mathcal{L})}$ is

$$\lambda_{\min} = \min_E b_E^{-1} \mathrm{ord}_E(D), \quad \text{and} \quad \lambda_{\max} = \max_E b_E^{-1} \mathrm{ord}_E(D) = \mathrm{ord}_{E_0}(D), \tag{4.42}$$

where E_0 is the strict transform of $X \times \{0\}$.

We will use this description to compute a special example, the so-called *Deformation to the normal cone*, in Section 7.2.1.

An important application of Duistermaat-Heckman measure is to compute the non-Archimedean Monge-Ampère energy of test configurations.

Lemma 4.1. For any $\varphi = \varphi_{(\mathcal{X},\mathcal{L})} \in \mathcal{H}^{\mathrm{NA}}(L)$, then we have

$$E^{\mathrm{NA}}(\varphi) = \int_{\mathbb{R}} \lambda dv, \tag{4.43}$$

where dv is the Duistermaat-Heckman measure of $(\mathcal{X}, \mathcal{L})$.

4.4 Non-Archimedean pluripotential theory

In this section, we briefly introduce the non-Archimedean pluripotential theory developed by Boucksom-Jonsson. We refer to [78].

Definition 4.5. A *psh metric* on L^{an} is a function $\varphi : X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$, not identically $-\infty$, that can be written as the limit of a decreasing sequence in $\mathcal{H}^{\mathrm{NA}}(L)$.

Denote by $\mathrm{Psh}^{\mathrm{NA}}(L)$ the set of non-Archimedean psh metrics on L^{an}. We endow $\mathrm{Psh}^{\mathrm{NA}}(L)$ with the topology of point-wise convergence on X^{div}, called *weak topology* of $\mathrm{Psh}^{\mathrm{NA}}(L)$.

Pick $(n+1)$ ample line bundle $L_i \in \mathrm{Amp}(X)$ and $\varphi_i \in \mathcal{H}^{\mathrm{NA}}(L_i)$, $i = 0, \cdots, n$, and choose a dominating test configuration $\mathcal{X}$ such that $\varphi = \varphi_{(\mathcal{X},\mathcal{L}_i)}$, we recall

Definition 4.6 ([78]Definition 3.12)**.** The *energy pairing* takes an $(n+1)$-tuple $\varphi_i = \varphi_{(\mathcal{X},\mathcal{L}_i)} \in \mathcal{H}^{\mathrm{NA}}(L_i)$, $i = 0, \cdots, n$ to

$$(L_0, \varphi_0) \cdot \cdots \cdot (L_n, \varphi_n) := (\overline{\mathcal{L}}_0 \cdot \cdots \cdot \overline{\mathcal{L}}_n) \in \mathbb{R}, \tag{4.44}$$

where $\overline{\mathcal{L}_i}$ on $\overline{\mathcal{X}}$ is the canonical compactification of $(\mathcal{X}, \mathcal{L}_i)$.

The original definition in [78] fits in a very general setting, like L_i and $\mathcal{L}_i$ may be not ample. But for simplicity, we just consider the ample case.

For any $(n+1)$-tuple $(L_i, \varphi_i) \in \mathrm{Amp}(X) \times \mathrm{Psh}^{\mathrm{NA}}(L_i)$, the following quantity

$$(L_0, \varphi_0) \cdot \cdots \cdot (L_n, \varphi_n) := \inf_{\psi_i \in \mathcal{H}^{\mathrm{NA}}(L_i), \psi_i \geqslant \varphi_i} (L_0, \psi_0) \cdot \cdots \cdot (L_n, \psi_n) \tag{4.45}$$

gives a unique extension of the energy pairing to $\prod_{i=0}^{n} \mathrm{Psh}^{\mathrm{NA}}(L_i)$, which satisfies

- upper semicontinuous with respect to weak topology on $\mathrm{Psh}^{\mathrm{NA}}(L_i)$,
- increasing in each variable,
- continuous along decreasing sequences.

Definition 4.7. The *Monge-Ampère energy* is defined by

$$E^{\mathrm{NA}}(\varphi) := \frac{(L, \varphi)^{n+1}}{(n+1)\mathrm{Vol}(L)} \tag{4.46}$$

for any $\varphi \in \mathrm{Psh}^{\mathrm{NA}}(L)$. The space of psh metrics of finite energy on L^{an} is defined as

$$\mathcal{E}^{1,\mathrm{NA}}(L) := \{\varphi \in \mathrm{Psh}^{\mathrm{NA}}(L) \mid E^{\mathrm{NA}}(\varphi) > -\infty\}. \tag{4.47}$$

We endow $\mathcal{E}^{1,\mathrm{NA}}(L)$ with the *strong topology*, which is the coarsest refinement of the weak topology of $\mathrm{Psh}^{\mathrm{NA}}(L)$ (i.e., the topology of point-wise convergence on X^{div}) in which $E^{\mathrm{NA}} : \mathcal{E}^{1,\mathrm{NA}}(L) \to \mathbb{R}$ is continuous.
Thus a sequence $\{\varphi_i\}$ in $\mathcal{E}^{1,\mathrm{NA}}(L)$ converges to $\varphi \in \mathcal{E}^{1,\mathrm{NA}}(L)$ if and only if $\varphi_i \to \varphi$ point-wise on X^{div}. Moreover, it converges strongly if and only if we further have $E^{\mathrm{NA}}(\varphi_i) \to E^{\mathrm{NA}}(\varphi)$.

By the continuity of the energy pairing along decreasing sequence, then for a decreasing sequence in $\mathcal{E}^{1,\mathrm{NA}}(L)$, the weak convergence is same as the strong convergence.

By the definition of non-Archimedean psh metrics, we have

Proposition 4.2. For any $\varphi \in \mathcal{E}^{1,\mathrm{NA}}(L)$, there exists a sequence $\varphi_i \in \mathcal{H}^{\mathrm{NA}}(L)$ such that φ_i converges strongly to φ. In other words, the space $\mathcal{H}^{\mathrm{NA}}(L)$ is dense in $\mathcal{E}^{1,\mathrm{NA}}(L)$ with respect to the strong topology.

For each n-tuple $(L_i, \varphi_i) \in \mathrm{Amp}(X) \times \mathcal{E}^{1,\mathrm{NA}}(L_i)$, there exists a unique probability measure

$$\mathrm{MA}^{\mathrm{NA}}(\varphi_1, \cdots, \varphi_n) \in C^0(X^{\mathrm{an}})^{\vee}, \tag{4.48}$$

called *mixed Monge-Ampère measure*, such that

$$\int_{X^{\mathrm{an}}} \varphi \mathrm{MA}^{\mathrm{NA}}(\varphi_1, \cdots, \varphi_n) := \frac{1}{(L_1 \cdot \cdots \cdot L_n)} (0, \varphi) \cdot (L_1, \varphi_1) \cdot \cdots \cdot (L_n, \varphi_n) \tag{4.49}$$

for any $\varphi \in \mathrm{PL}(X^{\mathrm{an}})$.

Indeed, by the density of $\mathrm{PL}(X^{\mathrm{an}})$ in $C^0(X^{\mathrm{an}})$ (see Theorem 4.1), the formula (4.49) uniquely determines the mixed Monge-Ampère measure (4.48). For any $\varphi = \varphi_D \in$

$\mathrm{PL}(X^{\mathrm{an}})$ with $D \in \mathrm{VCar}(\mathcal{X})_{\mathbb{Q}}$ (we may assume that $\mathcal{X}$ is dominating), we choose an ample test configuration $(\mathcal{X}, \mathcal{L}')$ for some (X, L') such that $\mathcal{L}' + D$ is relative ample. Then each $\varphi \in \mathrm{PL}(X^{\mathrm{an}})$ can be written as a difference of functions in $\mathcal{H}^{\mathrm{NA}}(L')$ for some ample line bundle L'. Hence, the right-hand side in (4.49) is well-defined. By the increasing of energy pairing, then $\varphi \mapsto (0, \varphi) \cdot (L_1, \varphi_1) \cdot \cdots \cdot (L_n, \varphi_n)$ is a positive linear form on $\mathrm{PL}(X^{\mathrm{an}})$. By density of $\mathrm{PL}(X^{\mathrm{an}})$ in $C^0(X^{\mathrm{an}})$, it uniquely extends to a positive linear form on $C^0(X^{\mathrm{an}})$. We conclude that the equality (4.49) gives rise to a positive measure on X^{an}.

When $\varphi = 1$, we have

$$\begin{aligned}
\int_{X^{\mathrm{an}}} \mathrm{MA}^{\mathrm{NA}}(\varphi_1, \cdots, \varphi_n) =& \frac{1}{(L_1 \cdot \cdots \cdot L_n)} (0,1) \cdot (L_1, \varphi_1) \cdot \cdots \cdot (L_n, \varphi_n) \\
=& \frac{1}{(L_1 \cdot \cdots \cdot L_n)} \inf_{\psi_i \in \mathcal{H}^{\mathrm{NA}}(L_i), \psi_i \geqslant \varphi_i} (0,1) \cdot (L_1, \psi_1) \cdot \cdots \cdot (L_n, \psi_n) \\
=& \frac{1}{(L_1 \cdot \cdots \cdot L_n)} \inf_{\psi_i \in \mathcal{H}^{\mathrm{NA}}(L_i), \psi_i \geqslant \varphi_i} [\mathcal{X}_0] \cdot \overline{\mathcal{L}}_1 \cdot \cdots \cdot \overline{\mathcal{L}}_n \\
=& \frac{1}{(L_1 \cdot \cdots \cdot L_n)} \inf_{\psi_i \in \mathcal{H}^{\mathrm{NA}}(L_i), \psi_i \geqslant \varphi_i} [\mathcal{X}_1] \cdot \overline{\mathcal{L}}_1 \cdot \cdots \cdot \overline{\mathcal{L}}_n \\
=& \frac{1}{(L_1 \cdot \cdots \cdot L_n)} \inf_{\psi_i \in \mathcal{H}^{\mathrm{NA}}(L_i), \psi_i \geqslant \varphi_i} (\overline{\mathcal{L}}_1|_X) \cdot \cdots \cdot (\overline{\mathcal{L}}_n|_X) \\
=& 1,
\end{aligned} \tag{4.50}$$

where the fourth equality holds by flatness of $\pi : \overline{\mathcal{X}} \to \mathbb{P}^1$. Thus, the equality (4.49) defines a probability measure.

In the special case, $\varphi_i = \varphi_{(\mathcal{X}, \mathcal{L}_i)} \in \mathcal{H}^{\mathrm{NA}}(L_i)$, $i = 1, \cdots, n$, and $\mathcal{X}_0 := \sum_j b_j E_j$, then

$$\mathrm{MA}^{\mathrm{NA}}(\varphi_1, \cdots, \varphi_n) = \sum_j b_j (\mathcal{L}_1|_E \cdot \cdots \cdot \mathcal{L}_n|_E) \delta_{v_j}, \tag{4.51}$$

where $v_j := b_j^{-1} r(\mathrm{ord}_{E_j})$ and $r : (X_{\mathbb{C}})^{\mathrm{div}} \to X^{\mathrm{div}}$ is the restriction map.

Definition 4.8. The non-Archimedean Monge-Amère measure of $\varphi \in \mathcal{E}^{1,\mathrm{NA}}(L)$ is defined as

$$\mathrm{MA}^{\mathrm{NA}}(\varphi) := \mathrm{Vol}(L)^{-1} \mathrm{MA}^{\mathrm{NA}}(\varphi, \cdots, \varphi). \tag{4.52}$$

We will write

$$\mathrm{MA}^{\mathrm{NA}}(\varphi_1^{[k_1]}, \cdots, \varphi_s^{[k_s]}) = \mathrm{Vol}(L)^{-1} \mathrm{MA}^{\mathrm{NA}}(\overbrace{\varphi, \cdots, \varphi}^{k_1}, \cdots, \overbrace{\varphi_s, \cdots \varphi_s}^{k_s}) \tag{4.53}$$

for any $\varphi_i \in \mathcal{E}^{1,\mathrm{NA}}(L)$, $i = 1, \cdots, s$.

With the help of the definition of the non-Archimedean mixed Monge-Amère measure, we can define the non-Archimedean functional on $\mathcal{E}^{1,\mathrm{NA}}(L)$ as the same formulation of Archimedean case. For any $\varphi \in \mathcal{E}^{1,\mathrm{NA}}(L)$, one defines

$$\Lambda^{\mathrm{NA}}(\varphi) := \int_{X^{\mathrm{an}}} \varphi \mathrm{MA}^{\mathrm{NA}}(0). \tag{4.54}$$

We can re-define

$$E^{\mathrm{NA}}(\varphi) := \frac{1}{n+1} \sum_{k=0}^{n} \int_{X^{\mathrm{an}}} \varphi \mathrm{MA}^{\mathrm{NA}}(\varphi^{[k]}, 0^{[n-k]}). \tag{4.55}$$

and

$$J^{\mathrm{NA}}(\varphi) := \Lambda^{\mathrm{NA}}(\varphi) - E^{\mathrm{NA}}(\varphi). \tag{4.56}$$

Let $(\mathcal{Y}, \mathcal{Q})$ be a test configuration of (X, Q), where $Q = \mathcal{Q}|_{X\times\{1\}}$, we write $\mathcal{Q} = \mathcal{Q}_1 - \mathcal{Q}_2$ with $\mathcal{Q}_i$ being relative semiample. Then one also defines:

$$\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{Q}}, \varphi_1, \cdots, \varphi_{n-1}) := \mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{Q}_1}, \varphi_1, \cdots, \varphi_{n-1}) - \mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{Q}_2}, \varphi_1, \cdots, \varphi_{n-1}). \tag{4.57}$$

We define

$$(E^{\mathcal{Q}})^{\mathrm{NA}}(\varphi) := \sum_{k=0}^{n-1} \int_{X^{\mathrm{an}}} \varphi \mathrm{MA}^{\mathrm{NA}}(\varphi_{\mathcal{Q}}, \varphi^{[k]}, 0^{[n-1-k]}). \tag{4.58}$$

When $(\mathcal{Y}, \mathcal{Q}) = (X_{\mathbb{C}} := X \times \mathbb{C}, K^{\log}_{X_{\mathbb{C}}/\mathbb{C}} = p_1^* K_X)$, we denote

$$R^{\mathrm{NA}}(\varphi) := (E^{K^{\log}_{X_{\mathbb{C}}/\mathbb{C}}})^{\mathrm{NA}}(\varphi), \tag{4.59}$$

called the *Ricci energy* of φ. In [83], the authors generalized the definition of log discrepency functional $A_X : X^{\mathrm{an}} \to [0, +\infty]$. We do not review their definition, but emphasis two important facts:

- $A = +\infty$ on $X^{\mathrm{an}} \setminus X^{\mathrm{val}}$ and $A < +\infty$ on X^{div}.
- A is lower semicontinuous (This is essential difficulty in the study of YTD conjecture).

For any $\phi \in \mathcal{E}^{1,\mathrm{NA}}$, one defines

$$H^{\mathrm{NA}}(\phi) := \int_{X^{\mathrm{an}}} A_X(x) \mathrm{MA}^{\mathrm{NA}}(\varphi), \tag{4.60}$$

and

$$M^{\mathrm{NA}} := H^{\mathrm{NA}} + R^{\mathrm{NA}} + \hat{S} E^{\mathrm{NA}}. \tag{4.61}$$

Since A_X is lower semicontinuous, then H^{NA} is in general not continuous, just lower semicontinuous, with respect to the strong topology of $\mathcal{E}^{1,\mathrm{NA}}(L)$.

In particular, when $\varphi \in \mathcal{H}^{\mathrm{NA}}(L)$, then the above those functional coincide with those defined by intersection numbers in Section 4.2.

By [78][Theorem 7.34], we summarize the continuity of non-Archimedean functional as follows and use it later.

Proposition 4.3. The functionals $(E^Q)^{\mathrm{NA}}$, J^{NA} are strongly continuous in $\mathcal{E}^{1,\mathrm{NA}}(L)$.

4.5 Psh rays and test configuration

In this subsection, we build a bridge between the Archimedean side and the non-Archimedean side by giving a relation between psh rays with algebraic singularities and test configurations. This is the important step to construct framework between Archimedean and non-Archimedean formulation in the cscK (in particular, Kähler-Einstein) problem in terms of the variational method.

For each psh ray $\Phi : \mathbb{R}_{>0} \to \mathrm{Psh}(X, \omega)$, $\sup_X \phi_t$ is a convex function on t. Then the slope at infinity

$$\lambda_{\max} := \lim_{t\to\infty} t^{-1} \sup_X \phi_t \tag{4.62}$$

exists in $\mathbb{R} \cup \{+\infty\}$. A psh ray $\Phi : \mathbb{R}_{>0} \to \mathrm{Psh}$ has *linear growth* if $\lambda_{\max} < \infty$, which is equivalent to that $\sup_X \phi_t = O(t)$.

For rays in $\mathcal{E}^1$, the condition (4.62) means $d_1(\phi_t, 0) = O(t)$ as $t \to \infty$. In particular, any psh geodesic ray has linear growth.

For any psh ray Φ with linear growth, $\Phi - at$ is bounded above as $t \to \infty$, for some $a \in \mathbb{R}$. Equivalently, the S^1-invariant $p_1^*\omega_0$-psh function Ψ on $X \times \mathbb{D}^*$, defined by

$$\Psi(x, \tau) := \phi_{-\log|\tau|}(x) + a \log|\tau|, \tag{4.63}$$

is bounded above near $X \times \{0\}$. Thus, it can uniquely extend to a quasi-psh function on $X \times \mathbb{D}$. For any divisorial valuation w on $X_{\mathbb{C}}$ (we always use notations w representing the valuation on $X_{\mathbb{C}}$ and v representing the valuation on X), $w(\Psi) \geqslant 0$ makes sense as a generic Lelong number on a suitable blow up. Concretely, if we write $w = \mathrm{ord}_E$ for some $E \subset W \xrightarrow{\rho} X_{\mathbb{C}}$, then we define

$$\begin{aligned} w(\Psi) :=& w(\rho^*\Psi) \\ =& \text{Lelong number of } \rho^*\Psi \text{ at the general point of } E \\ =& \inf_{x\in E} \nu(x, \rho^*\Psi), \end{aligned} \tag{4.64}$$

where $\nu(x, \rho^*\Psi)$ is the Lelong number of $\rho^*\Psi$ at x. We have used the well-known Siu's result for the third equality.

One can define

$$w(\Phi) := w(\Psi) - aw(\tau), \tag{4.65}$$

which is independent of the choice of a.

Definition 4.9. For any psh ray Φ with linear growth, one associates a function

$$\Phi_{\mathrm{NA}} : X_{\mathbb{Q}}^{\mathrm{div}} \to \mathbb{R} \tag{4.66}$$

by setting $\Phi_{\mathrm{NA}}(v) := -\sigma(v)(\Phi)$.

In fact, $\Phi_{\mathrm{NA}} \in \mathrm{Psh}^{\mathrm{NA}}(L)$ for any psh ray of linear growth (see [15][Theorem 6.2]).

Recall for trivial valuation v_{triv}, $\sigma(v_{\mathrm{triv}}) = \mathrm{ord}_{X\times\{0\}}$. It has the following property from definition (see [15][Lemma 4.3])

$$\Phi_{\mathrm{NA}}(v_{\mathrm{triv}}) = \sup_{X_{\mathbb{Q}}^{\mathrm{div}}} \Phi_{\mathrm{NA}} = \lambda_{\max}. \tag{4.67}$$

Choosing a smooth Hermitian metric h on L such that $\mathrm{Ric}(h) = \omega$, then one can setup a one-to-one correspondence

$$\left\{ \text{ psh ray } \Phi : \mathbb{R}_{\geqslant 0} \to \mathrm{Psh}(X, \omega) \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} S^1\text{-invariant psh metric} \\ e^{-\Phi} p_1^* h \text{ on } (X \times \mathbb{D}^*, p_1^* L) \end{array} \right\}. \tag{4.68}$$

We say that Φ induces a psh metric on a normal test configuration $(\mathcal{X}, \mathcal{L})$ if the corresponding psh metric on $(X \times \mathbb{D}^*, p_1^* L) \cong (\mathcal{X}, \mathcal{L})|_{\mathbb{D}^*}$ extends to a psh metric on $(\mathcal{X}, \mathcal{L})|_{\mathbb{D}}$ (psh function on the complex space Y means a psh function restricted from the ambient space. See Demailly's notes [84] for pluripotential theory on singular complex space).

The following lemma gives a characterization which psh ray induces a psh metric of a test configuration,

Lemma 4.2. ([15][Lemma 4.4]) Given a psh ray $\Phi : \mathbb{R}_{\geqslant 0} \to \mathrm{Psh}(X, \omega)$ and a normal test configuration $(\mathcal{X}, \mathcal{L})$, the following are equivalent:

(i) Φ induces a psh metric on $(\mathcal{X}, \mathcal{L})$

(ii) Φ has linear growth and $\Phi_{\mathrm{NA}} \leqslant \varphi_{(\mathcal{X}, \mathcal{L})}$.

If the induced psh metric in (i) is further locally bounded, then $\Phi_{\mathrm{NA}} = \varphi_{(\mathcal{X}, \mathcal{L})}$.

Definition 4.10. A psh ray Φ has algebraic singularities if it induces a locally bounded psh metric on some normal semiample test configuration $(\mathcal{X}, \mathcal{L})$.

By Lemma 4.2, thus such a psh ray Φ has linear growth and $\Phi_{\mathrm{NA}} = \varphi_{(\mathcal{X}, \mathcal{L})}$. If given a semiample test configuration, there exists those psh rays.

Lemma 4.3. For any $\varphi \in \mathcal{H}^{\mathrm{NA}}(L)$, there exists a smooth psh ray Φ with algebraic singularities such that $\Phi_{\mathrm{NA}} = \varphi$. Furthermore, for any psh ray Ψ with $\Psi_{\mathrm{NA}} \leqslant \varphi$, then

$\Psi \leqslant \Phi + O(1)$.

Remark 4.2. In Lemma 4.3, Φ is just psh ray. Psh ray with algebraic singularities and semiample test configuration are not a one-to-one correspondence. It is possible that there exist many psh rays with algebraic singularities on the same semiample test configuration.

In fact, if we take the envelope of such all psh ray with algebraic singularities, then get a $C^{1,1}$ psh geodesic ray with algebraic singularities. It is constructed by Phong and Sturm in terms of the approximation argument (see [85-86]), the so called *Phong-Sturm geodesic ray*. This type geodesic ray is one-to-one corresponding to semiample test configuration.

The following result connects the Archimedean side and non-Archimedean side.

Theorem 4.4 ([87]). For any $\varphi = \varphi_{(\mathcal{X},\mathcal{L})} \in \mathcal{H}^{\mathrm{NA}}(L)$, let Φ be any locally bounded S^1-invariant Hermitian metric on $\mathcal{L}$. Then for any $F \in \{E, \Lambda, J, \mathcal{J}, E^Q, R\}$ satisfies

$$F'^{\infty}(\Phi) = F^{\mathrm{NA}}(\varphi), \tag{4.69}$$

where

$$F'^{\infty}(\Phi) := \lim_{t\to\infty} t^{-1} F(\phi_t). \tag{4.70}$$

For $F \in \{H, M\}$, this identity holds if Φ is a smooth positively curved Hermitian metric on $\mathcal{L}$.

As stated in section 2.2.2.1, in general, there is not smooth geodesic ray. For the Phong-Sturm geodesic ray, entropy H and K-energy M don't satisfy the slope identity (4.69), just hold inequality

$$H'^{\infty}(\Phi) \geqslant H^{\mathrm{NA}}(\varphi), \quad M'^{\infty}(\Phi) \geqslant M^{\mathrm{NA}}(\varphi). \tag{4.71}$$

This is essential difficult of variational method in general cscK problem.

In [15], the authors introduced a more general geodesic ray, the so-called *maximal geodesic ray* and built a one-to-one correspondence between such geodesic rays and functions in $\mathcal{E}^{1,\mathrm{NA}}$.

Definition 4.11. A geodesic ray $\Psi : \mathbb{R}_{\geqslant 0} \to \mathcal{E}^1(X,\omega)$ is *maximal* if any psh ray of linear growth $\Phi : \mathbb{R}_{>0} \to \mathcal{E}^1(X,\omega)$ with $\lim_{s\to 0}\phi(s) \leqslant \Psi(0)$ and $\Phi_{\mathrm{NA}} \leqslant \Psi_{\mathrm{NA}}$ satisfies $\Phi \leqslant \Psi$.

A maximal geodesic ray Ψ is thus uniquely determined by $\Psi(0)$ and Ψ_{NA}. We denoted by $\mathcal{R}^1_{\max}(X,\omega)$ the space of maximal geodesic rays in $\mathcal{E}^1$ emanating from 0.

Theorem 4.5 ([15]Theorem 6.6). (*i*) For any psh ray Φ of linear growth, $\Phi_{\mathrm{NA}} \in$

$\mathcal{E}^{1,\mathrm{NA}}(L)$, and

$$E^{\mathrm{NA}}(\Phi_{\mathrm{NA}}) \geqslant E'^{\infty}(\Phi) > -\infty, \tag{4.72}$$

equality holds iff Φ is maximal.

(*ii*) For any $\phi \in \mathcal{E}^1$ and $\varphi \in \mathcal{E}^{1,\mathrm{NA}}(L)$, there exists a unique maximal geodesic ray Φ emanating from ϕ satisfying $\Phi_{\mathrm{NA}} = \varphi$.

Theorem 4.6 ([21][Proposition 2.40, Lemma 2.41, Theorem 5.3]). For any $\varphi \in \mathcal{E}^{1,\mathrm{NA}}(L)$, let $\Phi = \{\phi(s)\}$ be the associated maximal geodesic ray. Then

$$J'^{\infty}(\Phi) = J^{\mathrm{NA}}(\varphi), \quad (E_\chi)'^{\infty}(\Phi) = (E^Q)^{\mathrm{NA}}(\varphi), \tag{4.73}$$

and

$$H'^{\infty}(\Phi) \geqslant H^{\mathrm{NA}}(\varphi), \quad M'^{\infty}(\Phi) \geqslant M^{\mathrm{NA}}(\varphi), \tag{4.74}$$

where $\chi \in c_1(Q)$.

4.6 The theory of test curves

In this section, we review the theory of test curves, developed in [88], [89], [90]. Our notations follow [90].

For any $\phi \in \mathrm{Psh}(X,\omega)$, we associate the following notion of envelopes:

$$P[\phi] := \overset{*}{\sup}\{\psi \in \mathrm{Psh}(X,\omega) \mid \psi \leqslant 0, \psi \leqslant \phi + C \text{ for some } C \in \mathbb{R}\}, \tag{4.75}$$

and

$$P[\phi]_{\mathcal{I}} := \overset{*}{\sup}\{\psi \in \mathrm{Psh}(X,\omega) \mid \psi \leqslant 0, \mathcal{I}(k\psi) \subset \mathcal{I}(k\phi), \text{ for all } k \in \mathbb{N}\}, \tag{4.76}$$

where " * " denotes the upper semicontinuous regularization and $\mathcal{I}(\phi)$ is the multiplier ideal sheaf, locally generated by holomorphic function f such that $|f|^2 e^{-\phi}$ is integrable. The envelope $P[\phi]$ is studied in [91]. The envelope $P[\phi]_{\mathcal{I}}$ is studied in [89], [90]. A potential $\phi \in \mathrm{Psh}(X,\omega)$ is called a *model potential* if $\phi = P[\phi]$, and it is called a $\mathcal{I}$-*model potential* if $\phi = P[\phi]_{\mathcal{I}}$.

In fact, for any $\phi \in \mathrm{Psh}(X,\omega)$, $P[\phi]_{\mathcal{I}}$ is a model potential. Indeed, since $P[\phi]_{\mathcal{I}} \leqslant 0$, then $P[\phi]_{\mathcal{I}} \leqslant P[P[\phi]_{\mathcal{I}}]$ by the definition of $P[\cdot]$. On the other hand, we have $P[P[\phi]_{\mathcal{I}}] = \lim_{C\to\infty} P(P[\phi]_{\mathcal{I}} + C, 0)$, where

$$P(u,v) := \sup\{w \in \mathrm{Psh}(X,\omega) \mid w \leqslant \min(u,v)\} \tag{4.77}$$

for any $u, v \in \mathrm{Psh}(X,\omega)$. Then we obtain that $aP[\phi]_{\mathcal{I}}$ and $aP[P[\phi]_{\mathcal{I}}]$ have same multiplier ideal sheaves for all $a \geqslant 0$ since multiplier ideal sheaves are stable under taking increasing limits by the strong openness of multiplier ideal sheaves [92]. Hence $P[P[\phi]_{\mathcal{I}}] \leqslant P[\phi]_{\mathcal{I}}$. We have $P[P[\phi]_{\mathcal{I}}] = P[\phi]_{\mathcal{I}}$.

As a result, if ϕ is $\mathcal{I}$-model then it is automatically model, but not vice versa.

A useful characterization of $\mathcal{I}$-model potential is the following result,

Theorem 4.7 ([89] Theorem 1.4). Let (X, L) be a polarized manifold and ω be a Kähler metric in $c_1(L)$. Suppose T is a holomorphic vector bundle of rank s on X. Then for any $\phi \in \mathrm{Psh}(X, \omega)$, we have

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(k\phi)) = \frac{s}{n!}\int_X \omega^n_{P[\phi]_{\mathcal{I}}} \geqslant \frac{s}{n!}\int_X \omega^n_{\phi}. \tag{4.78}$$

Moreover, if $\int_X \omega^n_\phi > 0$, then equality holds iff ϕ is $\mathcal{I}$-model.

Remark 4.3. Later, Darvas-Xia [93] generalized this result to the case of pseudo-effective line bundles.

We denoted by Psh^{Model} (resp. $\mathrm{Psh}^{Model}_{\mathcal{I}}$) the space of model potentials (resp. $\mathcal{I}$-model potentials) in $\mathrm{Psh}(X, \omega)$.

Definition 4.12. A *test curve* is a map $\psi = \psi_\bullet : \mathbb{R} \to \mathrm{Psh}^{Model} \cup \{-\infty\}$ such that

(i) $\psi_\bullet$ is concave in $\bullet$.

(ii) ψ is upper semicontinuous as a function $X \times \mathbb{R} \to [-\infty, \infty)$.

(iii) $\lim_{r\to-\infty} \psi_r = 0$ in L^1.

(iv) $\psi_r = -\infty$ for r large enough.

Set $r^+ := \inf\{r \in \mathbb{R} \mid \psi_r = -\infty\}$. We say $\psi_\bullet$ is *normalized* if $r^+ = 0$. The test curve $\psi_\bullet$ is called *bounded* if $\psi_r = 0$ for r small enough. Let $r^- := \sup\{r \in \mathbb{R} \mid \psi_r = 0\}$ in this case.

Definition 4.13. The *Monge-Ampère energy* of a test curve $\psi_\bullet$ is defined as

$$\mathbf{E}(\psi_\bullet) := r^+ + \frac{1}{V}\int_{-\infty}^{r^+} \left(\int_X \omega^n_{\psi_r} - \int_X \omega^n\right) dr. \tag{4.79}$$

A test curve $\psi_\bullet$ is called *of finite energy* if $\mathbf{E}(\psi_\bullet) > -\infty$. We denote by $\mathcal{TC}^1(X, \omega)$ the set of finite energy test curves.

There is a natural relation between test curves and psh geodesic rays by the Legendre duality.

Definition 4.14. Let $\ell \in \mathcal{R}^1(X, \omega)$, the *Legendre transform* of ℓ is defined as

$$\hat{\ell}_r := \inf_{t\geqslant 0}(\ell_t - tr), \quad r \in \mathbb{R}. \tag{4.80}$$

Let $\psi_\bullet \in \mathcal{TC}^1(X,\omega)$, the *inverse Legendre transform* of $\psi_\bullet$ is defined as

$$\check{\psi}_r := \sup_{r\in\mathbb{R}}(\psi_r + tr), \quad t \geqslant 0. \tag{4.81}$$

The following result gives a one-to-one correspondence between geodesic rays and test curves of finite energy.

Theorem 4.8 ([89][Theorem 3.7]). The Legendre transform and inverse Legendre transform gives a bijection between $\mathcal{R}^1(X,\omega)$ and $\mathcal{TC}^1(X,\omega)$. For any $\ell \in \mathcal{R}^1(X,\omega)$, we have

$$\sup_X \ell_1 = r^+ \tag{4.82}$$

and

$$E'^\infty(\ell) = \mathbf{E}(\widehat{\ell}). \tag{4.83}$$

Moreover, under this correspondence, $\mathcal{R}^\infty$ corresponds to the set of bounded test curves. For any $\ell \in \mathcal{R}^\infty$, then $\inf_X \ell_1 = r^-$.

Definition 4.15. A test curve $\psi_\bullet$ is called an *$\mathcal{I}$-model test curve* if ψ_r is $\mathcal{I}$-model for any $r < r^+$. We denote by $\mathcal{TC}^1_{\mathcal{I}}(X,\omega)$ the set of $\mathcal{I}$-model test curves of finite energy.

Theorem 4.9 ([89][Theorem 3.7]). The Legendre transform and inverse Legendre transform gives a bijection between $\mathcal{R}^1_{\max}(X,\omega)$ and $\mathcal{TC}^1_{\mathcal{I}}(X,\omega)$.

Let χ be a real smooth $(1,1)$-form on X, for any $\psi_\bullet \in \mathcal{TC}^1(X,\omega)$, one defines the *$\chi$-twisted Monge-Ampère energy* as

$$\mathbf{E}_\chi(\psi_\bullet) := r^+\frac{n}{V}\int_X \omega^{n-1}\wedge\chi + \frac{n}{V}\int_{-\infty}^{r^+}\left(\int_X \omega_{\psi_r}^{n-1}\wedge\chi - \int_X \omega^{n-1}\wedge\chi\right)dr. \tag{4.84}$$

When $\chi = -\mathrm{Ric}(\omega)$, we call $\mathbf{E}_\chi(\psi_\bullet)$ the *Ricci energy* of $\psi_\bullet$.

In [90], the author computed the twisted Mong-Ampère energy of the Legendre transform of a maximal geodesic ray.

Theorem 4.10 ([90][Theorem 6.7]). For any $\ell \in \mathcal{R}^1_{\max}(X,\omega)$, then we have

$$\mathbf{E}_\chi(\widehat{\ell}_\bullet) = (E_\chi)'^\infty(\ell). \tag{4.85}$$

Example 4.2. Let $\mathscr{F}$ be a filtration of R and $r \in \mathbb{R}$, then one defines

$$\psi_r = \overset{*}{\sup_{k\in\mathbb{Z}_+}}\left(\frac{1}{k}\overset{*}{\sup}\left\{\log|s|^2_{h^k} : s \in \mathcal{F}^{kr}R_k,\ \sup|s|_{h^k} \leqslant 1\right\}\right), \tag{4.86}$$

where "$*$" is the upper semicontinuous regularization. By [89][Theorem 3.11], $\psi_\bullet$ is a $\mathcal{I}$-model test curve.

Chapter 5 Valuative stability

In this chapter, we introduce the valuative stability of Fano varieties and polarized varieties. Firstly, Let us fix some notations.

Let (X, L) be a polarized variety, let $\pi : Y \to X$ be a surjective birational morphism.

Definition 5.1. A prime divisor $F \subset Y$ for some birational model Y over X is called a *prime divisor over X*. Denote by $\mathrm{PDiv}_{/X}$ the set of all prime divisors over X.

One can view F as a divisorial valuation ord_F on X, defined on the function field of X. In particular, we can always assume that Y is smooth by taking a resolution of singularities. Since the information of the valuation associated to F, which we are interested in, does not change under the resolution of singularities, see [94][Remark 2.23].

Definition 5.2. For any $F \in \mathrm{PDiv}_{/X}$, the *log discrepancy* $A_X(F)$ is defined to be

$$A_X(F) := 1 + \mathrm{ord}_F(K_Y - \pi^* K_X). \tag{5.1}$$

For any effective $\mathbb{Q}$-divisor D such that $K_X + D$ is $\mathbb{Q}$-Cartier, the log discrepancy $A_{(X,D)}(F)$ is defined to be

$$A_{(X,D)}(F) := 1 + \mathrm{ord}_F(K_Y - \pi^*(K_X + D)). \tag{5.2}$$

Note that the log discrepancy is well-defined, since we always assume that the canonical divisor K_X is $\mathbb{Q}$-Cartier.

For any prime divisor F over X and $x \in \mathbb{R}_{\geqslant 0}$, one can define a subspace $H^0(X, mL - xF) \subset H^0(X, mL)$ by the identifications

$$H^0(X, mL - xF) := H^0(Y, m\pi^* L - xF) \subset H^0(Y, m\pi^* L) \cong H^0(X, mL). \tag{5.3}$$

Then we denote

$$\mathrm{Vol}(L - xF) = \mathrm{Vol}(\pi^* L - xF). \tag{5.4}$$

For simplicity, we always omit π^*.

5.1 Valuative criterion on Fano varieties

Let X be a $\mathbb{Q}$-Fano variety, which means that the anticanonical divisor $-K_X$ is ample $\mathbb{Q}$-Cartier divisor and $A_X(\cdot)$ is always positive on $\mathrm{PDiv}_{/X}$ (the so-called *klt singularity*).

Fujita [22] and Li [24] define a numerical invariant

$$\beta_X(F) = A_X(F) - \frac{1}{(-K_X)^n}\int_0^\infty \mathrm{Vol}(-K_X - xF)dt, \tag{5.5}$$

for all $F \in \mathrm{PDiv}_{/X}$. We also denote

$$S_X(E) := \frac{1}{(-K_X)^n}\int_0^\infty \mathrm{Vol}(-K_X - xF)dt. \tag{5.6}$$

Given a sufficiently divisible $m \in \mathbb{N}$, a divisor $D \sim_{\mathbb{Q}} -K_X$ is called an *m-basis type* $\mathbb{Q}$-*divisor* if there exists a basis $\{s_1, \cdots, s_{d_m}\}$ of $H^0(X, m(-K_X))$ such that

$$D = \frac{1}{mN_m}\sum_{i=1}^{d_m}(s_i = 0). \tag{5.7}$$

Set

$$\delta_m(X) := \min\{\mathrm{lct}(X; D) \mid D \sim_{\mathbb{Q}} -K_X \text{ is } m\text{-basis type divisor}\}, \tag{5.8}$$

where

$$\mathrm{lct}(X, D) := \max\{c \in \mathbb{R}_{\geqslant 0} \mid A_{(X,cD)} \geqslant 0\}. \tag{5.9}$$

In [23], the authors define a stability threshold

$$\delta(X) := \limsup_{m\to\infty} \delta_m(X). \tag{5.10}$$

In fact, it is shown in [95] that the limsup is limit and

$$\lim_{m\to\infty} \delta_m(X) = \inf_{F\in\mathrm{PDiv}_{/X}} \frac{A_X(F)}{S_X(F)}. \tag{5.11}$$

The way of computing $\delta(X)$ as the infimum of the log canonical thresholds for a special kind of *complements* is important both conceptually and computationally, as it connects to more birational geometry tool.

Theorem 5.1 ([22], [24], [23], [95], [25]). A $\mathbb{Q}$-Fano variety X is

(i) K-semistable if and only if $\beta_X(F) \geqslant 0$ for all $F \in \mathrm{PDiv}_{/X}$, if and only if $\delta \geqslant 1$;

(ii) K-stable if and only if $\beta_X(F) > 0$ for all $F \in \mathrm{PDiv}_{/X}$;

(iii) uniformly K-stable if and only if $\beta_X(F) \geqslant \varepsilon S_X(F)$ for some $\varepsilon > 0$ and all $F \in \mathrm{PDiv}_{/X}$, if and only if $\delta(X) > 1$.

Remark 5.1. (i) In the above definition of β_X and $\delta(X)$, one can also obtain a numerical invariant if one replaced $-K_X$ by a ample $\mathbb{Q}$-divisor L as follows,

$$\delta(L) := \inf_{F\in\mathrm{PDiv}_{/X}} \frac{A_X(F)\mathrm{Vol}(L)}{S_L(F)} \tag{5.12}$$

which is used to test the *Ding-stability* (see [22]).

(ii) Very recently, Liu-Xu-Zhuang [32] showed that K-stability is equivalent to uniform

K-stability.

(iii) When a Fano variety X is uniformly K-stable, then the automorphism group $\mathrm{Aut}(X)$ is discrete.

5.2 Valutaive stability of polarized varieties

In this section, we review the β-invariant defined in [34] and state the definition of valuative stability.

In [34], Dervan and Legendre computed the Donaldson-Futaki invariant of the test configuration associated to a dreamy divisor for a polarized variety and obtained a new numerical invariant, which generalizes Fujita's original β-invariant. Then they show that valuative stability for dreamy divisors is equivalent to K-stability for integral test configurations. Here an integral test configuration means that its central fiber is integral.

For any ample divisor L, one defines the *slope* of (X, L) to be

$$\mu(L) := \frac{-K_X \cdot L^{n-1}}{L^n}. \tag{5.13}$$

For any $F \in \mathrm{PDiv}_{/X}$, Dervan-Legendre defined

$$\beta_L(F) := A_X(F)\mathrm{Vol}(L) + n\mu(L)\int_0^{+\infty} \mathrm{Vol}(L - xF)dx + \int_0^{+\infty} \mathrm{Vol}'(L - xF).K_X dx, \tag{5.14}$$

where

$$\mathrm{Vol}(L - xF) := \mathrm{Vol}(\pi^* L - xF), \tag{5.15}$$

and

$$\mathrm{Vol}'(L - xF).K_X := \frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(\pi^* L - xF + t\pi^* K_X). \tag{5.16}$$

For simplicity, we always omit π^* in the above notations. It follows from Theorem 3.1 that the notation $\mathrm{Vol}'(L - xF).K_X$ is well-defined for any $L \in \mathrm{Big}(X)$ and $F \in \mathrm{PDiv}_{/X}$. It is straightforward that $\beta_L(\cdot)$ depends only on the numerical equivalence class of L.

There are three numerical invariants on the space of prime divisors over X. Roughly speaking, these can be viewed as norms. For any $F \in \mathrm{PDiv}_{/X}$, we set

$$S_L(F) := \int_0^{+\infty} \mathrm{Vol}(L - xF)dx, \tag{5.17}$$

and

$$j_L(F) := \mathrm{Vol}(L)\tau_L(F) - S_L(F). \tag{5.18}$$

where $\tau_L(F)$ is the *pseudo-effective threshold* of F with respect to L, defined by

$$\tau_L(F) := \sup\{x \in \mathbb{R} \mid \mathrm{Vol}(L - xF) > 0\}. \tag{5.19}$$

Note that our notation $S_L(F)$ is different from the usual one, which is equal to $S_L(F)/\mathrm{Vol}(L)$. But just for convenience, we use this notation.

Lemma 5.1. When L is ample, for any prime divisor F, $\tau_L(F)$, $S_L(F)$ and $j_L(F)$ have the following relations

$$\frac{1}{n+1}\mathrm{Vol}(L)\tau_L(F) \leqslant j_L(F) \leqslant \frac{n}{n+1}\mathrm{Vol}(L)\tau_L(F), \tag{5.20}$$

and

$$\frac{1}{n+1}\mathrm{Vol}(L)\tau_L(F) \leqslant S_L(F) \leqslant \frac{n}{n+1}\mathrm{Vol}(L)\tau_L(F). \tag{5.21}$$

The invariant $j_L(\cdot)$ can be viewed as a norm corresponding to non-Archimedean functional J^{NA} and $S_L(\cdot)$ corresponds to $I^{\mathrm{NA}} - J^{\mathrm{NA}}$, see [34][Section 2], [33] and [19][Section 7.2]. The proof of this lemma is essentially same as that of Fujita [96] in Fano $L = -K_X$ case, also see [95][Proposition 3.11].

Proof of Lemma 5.1 We only need to show (5.21). The first inequality of (5.21) is given by the concavity of the volume function, which gives

$$\mathrm{Vol}(L - xF) \geqslant \mathrm{Vol}(L)\left(\frac{x}{\tau_L(F)}\right)^n. \tag{5.22}$$

It follows that

$$S_L(F) \geqslant \frac{1}{n+1}\mathrm{Vol}(L)\tau_L(F). \tag{5.23}$$

The second inequality is proved in [96][Proposition 2.1] (In [96], $L = -K_X$, but this condition is not used in the proof). ∎

For any $L \in \mathrm{Amp}(X)$, we define two numerical invariants:

$$s(L) := \sup\{s \in \mathbb{R} | -K_X - sL \text{ is ample}\}, \tag{5.24}$$

and

$$\tilde{s}(L) := \inf\{s \in \mathbb{R} | K_X + sL \text{ is ample}\}. \tag{5.25}$$

By definitions of $s(L)$ and $\tilde{s}(L)$, we have $\mu(L) \geqslant s(L)$ and $\mu(L) \leqslant \tilde{s}(L)$. Indeed, if one assume that $-K_X - \mu(L)L$ is ample, then

$$\begin{aligned} 0 < (-K_X - \mu(L)L) \cdot L^{n-1} &= (-K_X \cdot L^{n-1} - \mu(L)L^n) \\ &= \left(\frac{-K_X \cdot L^{n-1}}{L^n} - \mu(L)\right)L^n. \end{aligned} \tag{5.26}$$

This leads to a contradiction. It follows that $\mu(L) \geqslant s(L)$. Another one is similar.

We state the following useful lemma (see [34][Corollary 3.11]), and use this lemma repeatedly later in the thesis,

Lemma 5.2. For any big divisor $L \in N^1(X)$ and any prime divisor F over X, we have

$$\int_0^{\tau_L(F)} n\langle (L-xF)^{n-1}\rangle \cdot L dx = (n+1)\int_0^{\tau_L(F)} \mathrm{Vol}(L-xF)dx. \tag{5.27}$$

Proof. Using the integration by part and Theorem 3.1 we compute

$$\begin{aligned}
\int_0^{\infty} n\langle (L-xF)^{n-1}\rangle \cdot L dx &= \int_0^{\infty} \frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(L-xF+tL)dx \\
&= \int_0^{\infty} \frac{d}{dt}\Big|_{t=0} \Big((1+t)^n \mathrm{Vol}(L-\frac{x}{1+t}F)\Big)\, dx \\
&= n\int_0^{\infty} \mathrm{Vol}(L-xF)dx + \int_0^{\infty} \frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(L-\frac{x}{1+t}F)dx \\
&\overset{s:=\frac{x}{1+t}}{=} n\int_0^{\infty} \mathrm{Vol}(L-xF)dx + \int_0^{\infty} \frac{d}{ds}\Big|_{s=x} \mathrm{Vol}(L-sF)(-x)dx \\
&= n\int_0^{\infty} \mathrm{Vol}(L-xF)dx + \int_0^{\infty} (-x)d\mathrm{Vol}(L-xF) \\
&= n\int_0^{\infty} \mathrm{Vol}(L-xF)dx + (-x)\mathrm{Vol}(L-xF)\Big|_{x=0}^{\infty} \\
&\quad + \int_0^{\infty} \mathrm{Vol}(L-xF)dx \\
&= (n+1)\int_0^{\infty} \mathrm{Vol}(L-xF)dx.
\end{aligned} \tag{5.28}$$

This completes the proof of Lemma 5.2. ∎

By this lemma, we can re-write β as

$$\begin{aligned}
\beta_L(F) =& A_X(F)\mathrm{Vol}(L) + (n\mu(L)-(n+1)s(L))S_L(F) \\
&- \int_0^{+\infty} \mathrm{Vol}'(L-xF).(-s(L)L-K_X)dx,
\end{aligned} \tag{5.29}$$

or

$$\begin{aligned}
\beta_L(F) =& A_X(F)\mathrm{Vol}(L) + (n\mu(L)-(n+1)\tilde{s}(L))S_L(F) \\
&+ \int_0^{+\infty} \mathrm{Vol}'(L-xF).(\tilde{s}(L)L+K_X)dx.
\end{aligned} \tag{5.30}$$

In fact, when $(X, L) = (X, -K_X)$, then we have

$$\begin{aligned}
&\beta_{-K_X}(F) \\
=& A_X(F)\mathrm{Vol}(-K_X) + n\int_0^{+\infty} \mathrm{Vol}(-K_X-xF)dx - \int_0^{+\infty} \mathrm{Vol}'(-K_X-xF).(-K_X)dx
\end{aligned}$$

$$
\begin{aligned}
&=A_X(F)\mathrm{Vol}(-K_X)+n\int_0^{+\infty}\mathrm{Vol}(-K_X-xF)dx-(n+1)\int_0^{+\infty}\mathrm{Vol}(-K_X-xF)dx\\
&=\mathrm{Vol}(-K_X)\beta_X(F), \qquad (5.31)
\end{aligned}
$$

where we have used Lemma 5.2 for the second equality. Thus the β_L-invariant is a generalization of Fujita's invariant to arbitrary polarized varieties.

Definition 5.3. For any $L \in \mathrm{Amp}(X)$, (X, L) is called

(i) *valuatively semistable* (resp. *over dreamy divisors*) if

$$\beta_L(F) \geqslant 0 \tag{5.32}$$

for any (resp. dreamy) prime divisor F over X;

(ii) *valuatively stable* (resp. *over dreamy divisors*) if

$$\beta_L(F) > 0 \tag{5.33}$$

for any non-trivial (resp. dreamy) prime divisor F over X, in which the non-trivial prime divisor F means that the divisorial valuation associated to F is non-trivial;

(iii) *uniformly valuatively stable* (resp. *over dreamy divisors*) if there exists an $\varepsilon_L > 0$ such that

$$\beta_L(F) \geqslant \varepsilon_L S_L(F) \tag{5.34}$$

for any (resp. dreamy) prime divisor F over X.

Remark 5.2. (i) Note that in [34], valuative stability means β_L satisfies the demanded inequality for all dreamy divisors (see [34][Definition 2.6]). If β_L is nonnegative for all prime divisors over X, it is called *strongly valuatively semistable* in [34].

(ii) In [34] the authors use the norm $j_L(\cdot)$ to define uniformly valuative stability. By Lemma 5.1, then j_L and S_L are equivalent.

In [34], the authors showed a partial equivalence of the valuative criterion.

Theorem 5.2 ([34]). K-stability with respect to integral test configurations $\Leftrightarrow$ valautive stability over dreamy divisors.

Dervan-Legendre [34] also gave a sufficient condition involving the δ-invariant (see Remark 5.1) of uniformly valuative stability.

Corollary 5.1. Suppose that

$$\frac{\delta(L)+n\mu(L)}{n+1}L+K_X \tag{5.35}$$

is effective, then (X, L) is uniformly valuatively stable.

Proof. We set

$$\tau(L) := \inf\{s \in \mathbb{R} \mid K_X + sL \text{ is psef}\}. \tag{5.36}$$

It follows that $\tau(L)L + K_X$ is psef since the psef cone is closed. By the definition of δ-invariant (see Remark 5.1), we have

$$A_X(F)\mathrm{Vol}(F) \geqslant \delta(L)S_L(F). \tag{5.37}$$

By Lemma 5.2, we can re-write

$$\begin{aligned}
\beta_L(F) =& A_X(F)\mathrm{Vol}(F) + n\mu(L)S_L(F) \\
&+ \int_0^\infty \mathrm{Vol}'(L - xF).(\tau(L)L + K_X - \tau(L)L)dx \\
=& A_X(F)\mathrm{Vol}(F) + n\mu(L)S_L(F) - (n+1)\tau(L)S_L(F) \\
&+ \int_0^\infty \mathrm{Vol}'(L - xF).(\tau(L)L + K_X)dx \\
\geqslant& (\delta(L) + n\mu(L) - (n+1)\tau(L))S_L(F) + \int_0^\infty \mathrm{Vol}'(L - xF).(\tau(L)L + K_X)dx,
\end{aligned} \tag{5.38}$$

where we have used (5.37) for the third inequality.

Since $\tau(L)L + K_X$ is psef, then there exists a sequence $\{D_j\}$ of effective divisors such that

$$[D_j] \to [\tau(L)L + K_X]. \tag{5.39}$$

Since the positive intersection product $\langle (L - xF)^{n-1} \rangle$ is 1-cycles, then we have

$$\langle (L - xF)^{n-1} \rangle \cdot D_j \to \langle (L - xF)^{n-1} \rangle \cdot (\tau(L)L + K_X). \tag{5.40}$$

On the other hand, $\langle (L - xF)^{n-1} \rangle$ intersecting with an effective divisor is non-negative since it can be computed by the restricted volume (see [71][Theorem B]). Thus, we have

$$\langle (L - xF)^{n-1} \rangle \cdot D_j \geqslant 0. \tag{5.41}$$

It follows that

$$\langle (L - xF)^{n-1} \rangle \cdot (\tau(L)L + K_X) \geqslant 0. \tag{5.42}$$

We obtain

$$\beta_L(F) \geqslant (\delta(L) + n\mu(L) - (n+1)\tau(L))S_L(F). \tag{5.43}$$

Since

$$\frac{\delta(L)+n\mu(L)}{n+1}L+K_X \tag{5.44}$$

is effective, then it is psef, i.e.,

$$\tau(L)<\frac{\delta(L)+n\mu(L)}{n+1}. \tag{5.45}$$

We obtain

$$\delta(L)+n\mu(L)-(n+1)\tau(L)>0. \tag{5.46}$$

Together with (5.43), then (X,L) is uniformly valuatively stable. ∎

In this paper, we are interested in the openness of uniformly valuative stability. Our main theorem is stated as follows,

Theorem 5.3 ([40][Theorem 1]). The uniformly valuative stability locus

$$\mathrm{UVs}:=\{[L]\in\mathrm{Amp}(X)\mid (X,L)\text{ is uniformly valuatively stable}\} \tag{5.47}$$

is an open subcone of the ample cone $\mathrm{Amp}(X)$.

Chapter 6 Proof of the main theorems

6.1 Openness of uniformly valuative stability

In this section, we give a proof of Theorem 5.3.

We first give a rough idea of setup: Fix an ample $\mathbb{R}$-divisor L, which is uniformly valuatively stable, and choose a constant $\varepsilon_L > 0$ such that

$$\beta_L(F) \geqslant \varepsilon_L S_L(F) \tag{6.1}$$

for any prime divisor F over X. Our goal is to show that there exists a small open neighbourhod U of L in $\mathrm{Amp}(X)$ such that, for any L' in U there is a constant $\delta_{L'} > 0$ satisfying

$$\beta_{L'}(F) \geqslant \delta_{L'} S_{L'}(F) \tag{6.2}$$

for all prime divisor F over X.

To define such an open neighbourhod of L, we fix any norm $\|\cdot\|$ on $N^1(X)$ and define an open subset

$$U_\varepsilon := \{L' \in \mathrm{Amp}(X) \mid \|L' - L\| < \varepsilon\}. \tag{6.3}$$

If necessary, we shrink this neighbourhod, i.e. shrink ε.

It suffices to prove following these two estimates

$$\beta_{L'}(F) - \beta_L(F) \geqslant -f(\varepsilon) S_{L'}(F) \tag{6.4}$$

and

$$S_L(F) \geqslant s^-(\varepsilon) S_{L'}(F) \tag{6.5}$$

for any prime divisor F over X, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $s^- : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions with $f(\varepsilon) \to 0$ and $s^-(\varepsilon) \to 1$ as $\varepsilon \to 0$. Indeed,

$$\begin{aligned}
\beta_{L'}(F) =& \beta_L(F) + \beta_{L'}(F) - \beta_L(F) \\
\geqslant& \varepsilon_L S_L(F) - f(\varepsilon) S_{L'}(F) \\
\geqslant& \left(\varepsilon_L s^-(\varepsilon) - f(\varepsilon)\right) S_{L'}(F).
\end{aligned} \tag{6.6}$$

Lemma 6.1. For any $L \in \mathrm{Amp}(X)$, there exists a small constant $\varepsilon > 0$, such that for any $L' \in U_\varepsilon$ satisfying the following inequality

$$s^-(\varepsilon) S_{L'}(F) \leqslant S_L(F) \leqslant s^+(\varepsilon) S_{L'}(F) \tag{6.7}$$

for any $F \in \mathrm{PDiv}_{/X}$, where $s^- : \mathbb{R}^+ \to \mathbb{R}^+$ and $s^+ : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions with $s^-(\varepsilon) \to 1$ and $s^+(\varepsilon) \to 1$ as $\varepsilon \to 0$. Moreover, $s^-(\varepsilon) < 1$ and $s^+(\varepsilon) > 1$.

Proof. For any $L' \in U_\varepsilon$, we write it as $L' = L + \varepsilon H$ for some $\mathbb{R}$-divisor H in $N^1(X)$. For any $s > 0$, we can write

$$L + \varepsilon H = \frac{1}{1+s}\left(L + s(L + \frac{(1+s)\varepsilon}{s}H)\right), \tag{6.8}$$

and set

$$L_s := L + \frac{(1+s)\varepsilon}{s}H. \tag{6.9}$$

Then by choosing s small enough (determined later), which depends on ε, we can assume that both $(1+s)L - L_s$ and $L_s - (1-s)L$ are big. Indeed,

$$(1+s)L - L_s = s\left(L - \frac{(1+s)\varepsilon}{s^2}H\right), \tag{6.10}$$

for instance, take $s = \varepsilon^{1/4}$, then $(1+s)L - L_s$ is big when ε is small.

Thus we have

$$\begin{aligned}
S_{L'}(F) =& \int_0^{+\infty} \mathrm{Vol}(L' - xF)dx \\
=&(1+s)^{-n}\int_0^{+\infty} \mathrm{Vol}(L + sL_s - (1+s)xF)dx \\
\geqslant&(1+s)^{-n}\int_0^{+\infty} \mathrm{Vol}(L + (s-s^2)L - (1+s)xF)dx \\
=&\left(\frac{1+s-s^2}{1+s}\right)^n \int_0^{+\infty} \mathrm{Vol}(L - \frac{1+s}{1+s-s^2}xF)dx \\
=&\left(\frac{1+s-s^2}{1+s}\right)^{n+1} S_L(F).
\end{aligned} \tag{6.11}$$

On the other hand, similarly, we have

$$S_{L'}(F) \leqslant \left(\frac{1+s+s^2}{1+s}\right)^{n+1} S_L(F). \tag{6.12}$$

By taking

$$s^-(\varepsilon) = \left(1 - \frac{\varepsilon^{1/2}}{1+\varepsilon^{1/4}}\right)^{n+1} \text{ and } s^+(\varepsilon) = \left(1 + \frac{\varepsilon^{1/2}}{1+\varepsilon^{1/4}}\right)^{n+1}, \tag{6.13}$$

we finish the proof of Lemma 6.1.

∎

Comparing with the openness of valuative stability in Fano case ([39], [36]), it follows from the definition formulated in Section 5.1 that Lemma 6.1 suffices to show the openness of original β-invariant.

But in the polarized case, the β-invariant is more complicated involving the derivative part. A main difficulty in this case is to control the difference of the derivative part in the expression of β-invariant for two nearby ample divisors. It is hard to control the difference for all prime divisors in general. In fact, we do not need to show the inequality (6.4) for any prime divisor F over X. By the definition of uniformly valuative stability, we introduce a subset of prime divisors over X as in the next definition, on which it is clearly sufficient to test uniformly valuative stability.

In addition, the log discrepancy has no control generally. By considering the derivative part of β-invariant together with the log discrepancy (see (6.15)), we obtain a partial control of β-invariant (see Theorem 6.1), which is enough to show our main theorem.

Definition 6.1. For any $L \in \mathrm{Amp}(X)$, let

$$\mathcal{D}_L^{\mathrm{ud}} := \{F \in \mathrm{PDiv}_{/X} \mid \beta_L(F) \leqslant C_L S_L(F)\}, \tag{6.14}$$

for some constant $C_L > 0$ (determined later).

It follows that we only need to prove the inequality (6.4) for any $F \in \mathcal{D}_{L'}^{\mathrm{ud}}$. Since when $F \notin \mathcal{D}_{L'}^{\mathrm{ud}}$, it automatically satisfies the condition of uniformly valuative stability.

Then for any $F \in \mathcal{D}_{L'}^{\mathrm{ud}}$, we have

$$\begin{aligned} &A_X(F)\mathrm{Vol}(L') + \int_0^{+\infty} n\langle (L' - xF)^{n-1}\rangle \cdot (K_X + \tilde{s}(L')L')dx \\ &\leqslant \left(C_{L'} - n\mu(L') + (n+1)\tilde{s}(L')\right) S_{L'}(F), \end{aligned} \tag{6.15}$$

where we have used the Lemma 5.2. Now we choose $C_{L'} > 0$ such that $C_{L'} - n\mu(L') + (n+1)\tilde{s}(L') \geqslant 0$.

We establish the technique theorem to show the main theorem 5.3.

Theorem 6.1 ([40][Theorem 10]). Given a divisor $L \in \mathrm{Amp}(X)$, there exists a constant $\varepsilon_0 > 0$ and a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\varepsilon\to 0} f(\varepsilon) = 0$, such that for any $0 < \varepsilon \leqslant \varepsilon_0$ and any $L' \in U_\varepsilon$, the inequality

$$\beta_{L'}(F) - \beta_L(F) \geqslant -f(\varepsilon)S_{L'}(F) \tag{6.16}$$

is satisfied for all $F \in \mathcal{D}_{L'}^{\mathrm{ud}}$. Moreover, the choice of f only depends on X and L.

We first show the estimate of the second term of β-invariant, i.e. μS.

Lemma 6.2. For any $L \in \mathrm{Amp}(X)$, there exists a constant $\varepsilon_0 > 0$ and a continuous function $h : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$, such that for any $0 < \varepsilon \leqslant \varepsilon_0$ and any $L' \in U_\varepsilon$, the inequality

$$n\mu(L')S_{L'}(F) - n\mu(L)S_L(F) \geqslant -h(\varepsilon)nS_{L'}(F) \tag{6.17}$$

is satisfied for all $F \in \mathrm{PDiv}_{/X}$. Moreover, the choice of h only depends on X and L.

Proof. For simplicity, we denote

$$s_{-}(n) := \left(1 - \frac{s^2}{1+s}\right)^n < 1 \text{ and } s_{+}(n) := \left(1 + \frac{s^2}{1+s}\right)^n > 1. \tag{6.18}$$

For any $L' \in U_\varepsilon$, we can write

$$L' = L + \varepsilon H = \frac{1}{1+s}\left(L + sL_s\right) \tag{6.19}$$

in the same way as the proof of Lemma 6.1, for some $\mathbb{R}$-divisor H and L_s in $N^1(X)$. Thus we have

$$\begin{aligned}\mathrm{Vol}(L') =& (1+s)^{-n}\mathrm{Vol}(L + sL_s)\\ \geqslant& (1+s)^{-n}\mathrm{Vol}(L + (s - s^2)L)\\ =& s_{-}(n)\mathrm{Vol}(L).\end{aligned} \tag{6.20}$$

Similarly, one obtains

$$\mathrm{Vol}(L') \leqslant s_{+}(n)\mathrm{Vol}(L). \tag{6.21}$$

The proof falls naturally into two cases.

(1) When $\mu(L') \geqslant 0$, then we compute

$$\begin{aligned}&n\mu(L')S_{L'}(F) - n\mu(L)S_L(F)\\ \geqslant& nS_L(F)(s^{-}(\varepsilon)\mu(L') - \mu(L))\\ =& nS_L(F)\left((s^{-}(\varepsilon) - 1)\mu(L') + \frac{-K_X \cdot (L')^{n-1}}{\mathrm{Vol}(L')} - \frac{-K_X \cdot L^{n-1}}{\mathrm{Vol}(L)}\right)\\ \geqslant& nS_L(F)\left((s^{-}(\varepsilon) - 1)\mu(L') + \frac{-K_X \cdot (L')^{n-1}}{s_{+}(n)\mathrm{Vol}(L)} - \frac{-K_X \cdot L^{n-1}}{\mathrm{Vol}(L)}\right)\\ \geqslant& nS_L(F)\left((s^{-}(\varepsilon) - 1)\mu(L') + (s_{+}(n)^{-1} - 1)\frac{-K_X \cdot (L')^{n-1}}{\mathrm{Vol}(L)}\right.\\ &\left.+\frac{1}{\mathrm{Vol}(L)}(-K_X \cdot (L')^{n-1} - (-K_X)\cdot L^{n-1})\right)\\ \geqslant& nS_L(F)\left((s^{-}(\varepsilon) - 1)\mu(L') + (\frac{1}{s_{+}(n)} - 1)s_{+}(n)\mu(L')\right.\\ &\left.+\frac{1}{\mathrm{Vol}(L)}((-K_X)\cdot \varepsilon H((L')^{n-2} + (L')^{n-3}\cdot L + \cdots + L^{n-2})\right)\\ \geqslant& nS_L(F)\Big((s^{-}(\varepsilon) - s_{+}(n))\mu(L')\\ &+\varepsilon\frac{1}{\mathrm{Vol}(L)}((-K_X)\cdot H((L')^{n-2} + (L')^{n-3}\cdot L + \cdots + L^{n-2})\Big).\end{aligned} \tag{6.22}$$

In general, we do not know the sign of

$$\frac{1}{\mathrm{Vol}(L)}((-K_X)\cdot H((L')^{n-2}+(L')^{n-3}\cdot L+\cdots+L^{n-2}). \tag{6.23}$$

But we can cancel it directly if it is nonnegative. Therefore, without loss of generality, we may assume that it is negative. Then

$$\begin{aligned} n\mu(L')S_{L'}(F)-n\mu(L)S_L(F) \geqslant & (-h_1(\varepsilon)-g(\varepsilon))nS_L(F) \\ \geqslant & (-h_1(\varepsilon)-g(\varepsilon))s^-(\varepsilon)^{-1}nS_{L'}(F), \end{aligned} \tag{6.24}$$

where

$$g(\varepsilon)=-\varepsilon\frac{1}{\mathrm{Vol}(L)}((-K_X)\cdot H((L')^{n-2}+(L')^{n-3}\cdot L+\cdots+L^{n-2}) \tag{6.25}$$

which is a polynomial in ε with degree $n-1$ and $g(0)=0$, whose coefficients depend on $-K_X$, L, H, and the leading term is $\mathrm{Vol}(L)^{-1}(-K_X)\cdot H^{n-1}$, and $h_1:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function with $h_1(\varepsilon)\to 0$ as $\varepsilon\to 0$, which depends on $\mu(L')$.

In fact, h_1 is independent of the choice of L'. Since g is a polynomial with degree $n-1$ and L' can be represented by a basis of Nef cone (see the following Lemma 6.3), then the choice of g only depends on X and L.

(2) When $\mu(L')\leqslant 0$, the computation is similar. We omit it.

This completes the proof of Lemma 6.2 by taking $h=(h_1+g)s^-(\varepsilon)^{-1}$. ∎

Lemma 6.3. There exists a constant $a>0$, which depends on ε and L, such that

$$(1-a)L\leqslant L'\leqslant(1+a)L \tag{6.26}$$

for any $L'\in U_\varepsilon$. Moreover, such a can be chosen as small as we wish by choosing ε small.

Proof. For any $L'\in U_\varepsilon$, we write it as $L'=L+H$ for some $\mathbb{R}$-divisor H in $N^1(X)$ with $\|H\|<\varepsilon$. Set $\rho:=\dim_{\mathbb{R}}N^1(X)$. Since L is ample, there exists a basis $(A_1,\cdots,A_\rho)$ of $N^1(X)$ with each A_i in Nef(X), and there exists some $t_1,\cdots,t_\rho\in\mathbb{R}_{>0}$ such that $L=\sum_{i=1}^\rho t_iA_i$ with $\sum_{i=1}^\rho t_i=1$. Set $t_0=\min_i t_i\in\mathbb{R}_{>0}$. We may assume that the norm $\|\cdot\|$ is given by

$$\left\|\sum_{i=1}^{\rho}s_iA_i\right\|:=\sum_{i=1}^{\rho}|s_i|. \tag{6.27}$$

Set $H=\sum_{i=1}^{\rho} r_iA_i$ with $\|H\|<\varepsilon$ (i.e. $\sum_{i=1}^{\rho}|r_i|<\varepsilon$). Then we have

$$L'=L+H=\sum_{i=1}^{\rho}(t_i+r_i)A_i<\sum_{i=1}^{\rho}(t_i+\varepsilon)A_i=\sum_{i=1}^{\rho}(t_i+\frac{\varepsilon}{t_0}t_0)A_i\leqslant(1+\frac{\varepsilon}{t_0})\sum_{i=1}^{\rho}t_iA_i, \tag{6.28}$$

also

$$L'=L+H=\sum_{i=1}^{\rho}(t_i+r_i)A_i>\sum_{i=1}^{\rho}(t_i-\varepsilon)A_i=\sum_{i=1}^{\rho}(t_i-\frac{\varepsilon}{t_0}t_0)A_i\geqslant(1-\frac{\varepsilon}{t_0})\sum_{i=1}^{\rho}t_iA_i. \tag{6.29}$$

The proof is completed by taking $a=\varepsilon/t_0$, where $t_0=\min_i t_i$. ∎

Remark 6.1. Consistent with the notation in Section 3.2, " $\leqslant$ " means that their difference is a psef class. In fact, $(1+a)L-L'$ and $L'-(1-a)L$ are nef according to the proof of Lemma 6.3.

We now turn to the proof of Theorem 6.1.

Proof of Theorem 6.1 For any $L'\in U_\varepsilon$, we can write

$$L'=L+\varepsilon H=\frac{1}{1+s}\left(L+sL_s\right) \tag{6.30}$$

in the same way as the proof of Lemma 6.1, for some $\mathbb{R}$-divisor H and L_s in $N^1(X)$ such that $(1+s)L-L_s$ and $L_s-(1-s)L$ are big when ε is small enough, where $s=\varepsilon^{1/4}$.

We divide into following these two cases,

(1) One assume $\mu(L')\geqslant 0$, then $\tilde{s}(L')\geqslant 0$.

$$\begin{aligned}
&\beta_{L'}(F)-\beta_L(F)\\
=&A_X(F)(\mathrm{Vol}(L')-\mathrm{Vol}(L))+n\mu(L')S_{L'}(F)-n\mu(L)S_L(F)\\
&+\int_0^{+\infty}n\langle(L'-xF)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')dx-\tilde{s}(L')\int_0^{+\infty}n\langle(L'-xF)^{n-1}\rangle\cdot L'dx\\
&-\int_0^{+\infty}n\langle(L-xF)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')dx\\
&+\int_0^{+\infty}n\langle(L-xF)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')dx\\
&-\int_0^{+\infty}n\langle(L-xF)^{n-1}\rangle\cdot K_Xdx\\
\geqslant&n\mu(L')S_{L'}(F)-n\mu(L)S_L(F)+A_X(F)\mathrm{Vol}(L')(1-s_(n)^{-1})\\
&-\tilde{s}(L')\int_0^{+\infty}n\langle(L'-xF)^{n-1}\rangle\cdot L'dx+\tilde{s}(L')\int_0^{+\infty}n\langle(L-xF)^{n-1}\rangle\cdot L'dx
\end{aligned}$$

$$+\int_0^{+\infty} n\left(\langle (L'-xF)^{n-1}\rangle - \langle (L-xF)^{n-1}\rangle\right)\cdot(K_X+\tilde{s}(L')L')dx. \tag{6.31}$$

By Lemma 6.3, we can take a small positive constant a (recall $a=\varepsilon/t_0$) such that

$$(1-a)L \leqslant L' \leqslant (1+a)L, \tag{6.32}$$

for any $L'\in U_\varepsilon$. Then one obtains

$$(1-a)L-xF \leqslant L'-xF \leqslant (1+a)L-xF. \tag{6.33}$$

Therefore, by the continuity and homogeneity of the positive intersection product (see Proposition 3.1 or [71][Proposition 2.9]), we have

$$(1-a)^{n-1}\langle (L-\frac{x}{1-a}F)^{n-1}\rangle \leqslant \langle (L'-xF)^{n-1}\rangle \leqslant (1+a)^{n-1}\langle (L-\frac{x}{1+a}F)^{n-1}\rangle. \tag{6.34}$$

Since $K_X+\tilde{s}(L')L'$ is nef, we have

$$\begin{aligned}
&(1-a)^{n-1}\langle (L-\frac{x}{1-a}F)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')\\
\leqslant&\langle (L'-xF)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')\\
\leqslant&(1+a)^{n-1}\langle (L-\frac{x}{1+a}F)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L').
\end{aligned} \tag{6.35}$$

It follows that

$$\begin{aligned}
&\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')dx\\
\geqslant&(1-a)^{n-1}\int_0^{+\infty} n\langle (L-\frac{x}{1-a}F)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')dx\\
=&(1-a)^{n}\int_0^{+\infty} n\langle (L-xF)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')dx.
\end{aligned} \tag{6.36}$$

Thus, we obtain

$$\begin{aligned}
&\int_0^{+\infty} n\left(\langle (L'-xF)^{n-1}\rangle - \langle (L-xF)^{n-1}\rangle\right)\cdot(K_X+\tilde{s}(L')L')dx\\
\geqslant&(1-(1-a)^{-n})\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')dx.
\end{aligned} \tag{6.37}$$

Recall $s=\varepsilon^{1/4}$, when we choose ε small enough, then a ($=\varepsilon/t_0$, see Lemma 6.3) can be chosen small enough, such that

$$1-s\_(n)^{-1} \leqslant 1-(1-a)^{-n}. \tag{6.38}$$

Then, we obtain

$$A_X(F)\mathrm{Vol}(L')(1-s\_(n)^{-1})$$

$$
\begin{aligned}
&+\int_0^{+\infty} n\left(\langle (L'-xF)^{n-1}\rangle - \langle (L-xF)^{n-1}\rangle\right)\cdot(K_X+\tilde{s}(L')L')dx \\
&\geqslant (1-s_-(n)^{-1})\left(A_X(F)\mathrm{Vol}(L')+\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle\cdot(K_X+\tilde{s}(L')L')dx\right) \\
&\geqslant (1-s_-(n)^{-1})\left(C_{L'}-n\mu(L')+(n+1)\tilde{s}(L')\right)S_{L'}(F) \qquad (6.39)
\end{aligned}
$$

where we have used (6.15) and Lemma 5.2 for the second inequality.

Since L' is ample, by (6.34), we have

$$
\begin{aligned}
(1-a)^{n-1}\langle (L-\frac{x}{1-a}F)^{n-1}\rangle\cdot L' &\leqslant \langle (L'-xF)^{n-1}\rangle\cdot L' \\
&\leqslant (1+a)^{n-1}\langle (L-\frac{x}{1+a}F)^{n-1}\rangle\cdot L'. \qquad (6.40)
\end{aligned}
$$

Then one obtains

$$
\begin{aligned}
\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle\cdot L'dx &\leqslant \int_0^{+\infty}(1+a)^{n-1}n\langle (L-\frac{x}{1+a}F)^{n-1}\rangle\cdot L'dx \\
&=(1+a)^n\int_0^{+\infty} n\langle (L-xF)^{n-1}\rangle\cdot L'dx. \qquad (6.41)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\tilde{s}(L')\int_0^{+\infty} n\langle (L-xF)^{n-1}\rangle\cdot L'dx-\tilde{s}(L')\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle\cdot L'dx \\
&\geqslant \tilde{s}(L')((1+a)^{-n}-1)\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle\cdot L'dx \\
&=\tilde{s}(L')((1+a)^{-n}-1)(n+1)S_{L'}(F). \qquad (6.42)
\end{aligned}
$$

Note that here we have used $\tilde{s}(L')\geqslant 0$ and Lemma 5.2.

Combining (6.31), (6.39), (6.42), and (6.17) , we have

$$
\begin{aligned}
&\beta_{L'}(F)-\beta_L(F) \\
&\geqslant -h(\varepsilon)nS_{L'}(F)+\Big((1-s_-(n)^{-1})\big(C_{L'}-n\mu(L')+(n+1)\tilde{s}(L')\big) \\
&\quad +(n+1)\tilde{s}(L')((1+a)^{-n}-1)\Big)S_{L'}(F) \\
&\geqslant -f(\varepsilon)S_{L'}(F), \qquad (6.43)
\end{aligned}
$$

where $f:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function with $f(\varepsilon)\to 0$ as $\varepsilon\to 0$, which depends on $\mu(L')$, $\tilde{s}(L')$ and $C_{L'}$ and intersection numbers $(-K_X)\cdot(L')^k\cdot L^{n-1-k}$ for $k=0,\cdots,n-1$.

By definition, we know that $\mu(L')$ and $\tilde{s}(L')$ are continuous with respect to L'. Thus, we can choose $C_{L'}$ continuously depending on L'. Therefore, the choice of f only depends on X and L.

(2) One assumes $\mu(L') \leqslant 0$, then $s(L') \leqslant 0$. We use the same idea of case ((1)).

$$
\begin{aligned}
&\beta_{L'}(F) - \beta_L(F) \\
=&A_X(F)(\mathrm{Vol}(L') - \mathrm{Vol}(L)) + n\mu(L')S_{L'}(F) - n\mu(L)S_L(F) \\
&+ \int_0^{+\infty} n\langle (L' - xF)^{n-1}\rangle \cdot K_X dx + \int_0^{+\infty} n\langle (L' - xF)^{n-1}\rangle \cdot (-K_X - s(L')L')dx \\
&- \int_0^{+\infty} n\langle (L' - xF)^{n-1}\rangle \cdot (-K_X - s(L')L')dx \\
&+ \int_0^{+\infty} n\langle (L - xF)^{n-1}\rangle \cdot (-K_X - s(L')L')dx + s(L')\int_0^{+\infty} n\langle (L - xF)^{n-1}\rangle \cdot L'dx \\
\geqslant& n\mu(L')S_{L'}(F) - n\mu(L)S_L(F) + A_X(F)\mathrm{Vol}(L')(1 - s_-(n)^{-1}) \\
&- s(L')\int_0^{+\infty} n\Big(\langle (L' - xF)^{n-1}\rangle - \langle (L - xF)^{n-1}\rangle\Big) \cdot L'dx \\
&+ \int_0^{+\infty} n\left(\langle (L - xF)^{n-1}\rangle - \langle (L' - xF)^{n-1}\rangle\right) \cdot (-K_X - s(L')L')dx. \qquad (6.44)
\end{aligned}
$$

By (6.34) and Lemma 5.2, we have

$$
\begin{aligned}
&\int_0^{+\infty} n\left(\langle (L - xF)^{n-1}\rangle - \langle (L' - xF)^{n-1}\rangle\right) \cdot (-K_X - s(L')L')dx \\
\geqslant&((1+a)^{-n} - 1)\int_0^{+\infty} n\langle (L' - xF)^{n-1}\rangle \cdot (-K_X - s(L')L')dx \\
=&(1 - (1+a)^{-n})\int_0^{+\infty} n\langle (L' - xF)^{n-1}\rangle \cdot K_X \\
&+ ((1+a)^{-n} - 1)(-s(L'))(n+1)S_{L'}(F). \qquad (6.45)
\end{aligned}
$$

Since F belongs to $\mathcal{D}^{\mathrm{ud}}_{L'}$, one can obtain

$$
\begin{aligned}
A_X(F)\mathrm{Vol}(L')(1 - s_-(n)^{-1}) \geqslant&(1 - s_-(n)^{-1})(C_{L'} - n\mu(L'))S_{L'}(F) \\
&- (1 - s_-(n)^{-1})\int_0^{+\infty} n\langle (L' - xF)^{n-1}\rangle \cdot K_X dx. \qquad (6.46)
\end{aligned}
$$

Since L' is ample, by (6.34) and Lemma 5.2, we obtain

$$
\begin{aligned}
&(-s(L'))\int_0^{+\infty} n\Big(\langle (L' - xF)^{n-1}\rangle - \langle (L - xF)^{n-1}\rangle\Big) \cdot L'dx \\
\geqslant&(-s(L'))(1 - (1-a)^{-n})\int_0^{+\infty} n\langle (L' - xF)^{n-1}\rangle \cdot L'dx \\
=&(-s(L'))(1 - (1-a)^{-n})(n+1)S_{L'}(F). \qquad (6.47)
\end{aligned}
$$

In addition, we have the following natural lower bound,

$$\begin{aligned}
&\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle \cdot K_X dx \\
=&\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle \cdot (K_X+\tilde{s}(L')L')dx - \tilde{s}(L')\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle \cdot L' dx \\
\geqslant& -\tilde{s}(L')(n+1)S_{L'}(F).
\end{aligned} \tag{6.48}$$

Combining (6.17) and (6.44)-(6.48), we have

$$\begin{aligned}
&\beta_{L'}(F)-\beta_L(F) \\
\geqslant& -h(\varepsilon)nS_{L'}(F)+\Big((1-s_-(n)^{-1})(C_{L'}-n\mu(L')) \\
&+((1+a)^{-n}-1)(-s(L'))(n+1)+(-s(L'))(1-(1-a)^{-n})(n+1)\Big)S_{L'}(F) \\
&+\Big(1-(1+a)^{-n}+s_-(n)^{-1}-1\Big)\int_0^{+\infty} n\langle (L'-xF)^{n-1}\rangle \cdot K_X dx \\
\geqslant& -h(\varepsilon)nS_{L'}(F)+\Big((1-s_-(n)^{-1})(C_{L'}-n\mu(L')) \\
&+((1+a)^{-n}-(1-a)^{-n})(-s(L'))(n+1)\Big)S_{L'}(F) \\
&+\Big(s_-(n)^{-1}-(1+a)^{-n}\Big)(-\tilde{s}(L'))(n+1)S_{L'}(F) \\
\geqslant& -f(\varepsilon)S_{L'}(F).
\end{aligned} \tag{6.49}$$

where $f:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function with $f(\varepsilon)\to 0$ as $\varepsilon\to 0$, which depends on $\mu(L')$, $\tilde{s}(L')$, $s(L')$ and $C_{L'}$ and intersection numbers $(-K_X)\cdot(L')^k\cdot L^{n-1-k}$ for $k=0,\cdots,n-1$.

Similar to case (1), we can choose a continuous function f which only depends on X and L.

By combining above these two cases, we complete the proof of Theorem 6.1. ∎

Finally, we finish the proof of the main Theorem.

Proof of Theorem 5.3 For any $L\in \mathrm{UV}s$, by Theorem 6.1, there exists a constant $\varepsilon_0>0$ and a continuous function $f:\mathbb{R}^+\to\mathbb{R}^+$ with $\lim_{\varepsilon\to 0} f(\varepsilon)=0$, which only depends X and L, such that for any $0<\varepsilon\leqslant\varepsilon_0$ and any $L'\in U_\varepsilon$, the inequality

$$\beta_{L'}(F)-\beta_L(F)\geqslant -f(\varepsilon)S_{L'}(F) \tag{6.50}$$

is satisfied for all $F\in\mathcal{D}^{\mathrm{ud}}_{L'}$. Since L is uniformly valuatively stable, combining with (6.6), we have

$$\beta_{L'}(F)\geqslant\left(\frac{\varepsilon_L}{2}-f(\varepsilon)\right)S_{L'}(F), \tag{6.51}$$

for all F in $\mathcal{D}^{\mathrm{ud}}_{L'}$. It follows that there exists an $\varepsilon_0 > 0$ such that

$$\frac{\varepsilon_L}{2} - f(\varepsilon) > 0 \tag{6.52}$$

for all $L \in U_\varepsilon$ and any $0 < \varepsilon \leqslant \varepsilon_0$, Then for any $L \in U_\varepsilon$, we have

$$\beta_{L'}(F) \geqslant \varepsilon_{L'} S_{L'}(F) \tag{6.53}$$

for some constant $\varepsilon_{L'} > 0$ and all $F \in \mathcal{D}^{\mathrm{ud}}_{L'}$. Thus, L' belongs to UVs for any $L' \in U_\varepsilon$.

Finally, by definitions of β and S-invariant, we have

$$\beta_{kL'}(F) = k^n \beta_{L'}(F), \text{ and } S_{kL'}(F) = k^{n+1} S_{L'}(F), \tag{6.54}$$

for $k > 0$. Then $\mathbb{R}_+ U_\varepsilon \subset$ UVs. Therefore, the uniformly valuative stability locus UVs is an open subcone of $\mathrm{Amp}(X)$. ∎

6.2 Uniformly valuative stability threshold

As an immediate application of Theorem 6.1 and 5.3, in this section, we show the continuity of the uniformly valuative stability threshold.

Definition 6.2. For any $L \in \mathrm{Amp}(X)$, the *uniformly valuative stability threshold* of L is defined to be

$$\zeta(L) := \sup\{x \in \mathbb{R} \mid \beta_L(F) \geqslant x S_L(F) \text{ for any } F \in \mathrm{PDiv}_{/X}\}. \tag{6.55}$$

In fact, when $(X, L) = (X, -K_X)$ is Fano, we have $\zeta(L) = \delta(X) - 1$. This is the main motivation to study the ζ-invariant.

Recall the definition of δ-invariant, due to Blum and Jonsson [95],

$$\delta(L) = \inf_{F \in \mathrm{PDiv}_{/X}} \frac{A_X(F)\mathrm{Vol}(L)}{S_L(F)}. \tag{6.56}$$

Thus, one obtains

$$A_X(F)\mathrm{Vol}(L) \geqslant \delta(L) S_L(F) \tag{6.57}$$

for any F in $\mathrm{PDiv}_{/X}$. By (6.48), we have a natural lower bound

$$\beta_L(F) \geqslant (\delta(L) + n\mu(L) - (n+1)\tilde{s}(L)) S_L(F), \tag{6.58}$$

i.e.

$$\zeta(L) \geqslant \delta(L) + n\mu(L) - (n+1)\tilde{s}(L). \tag{6.59}$$

One can take a $c_L > 0$ such that $\delta(L) + n\mu(L) - (n+1)\tilde{s}(L) + c_L > 0$. Thus now we set

$C_L := \delta(L) + n\mu(L) - (n+1)\tilde{s}(L) + c_L > 0$ in the definition of $\mathcal{D}_L^{\mathrm{ud}}$. We also define

$$\zeta^{\mathrm{ud}}(L) := \sup\{x \in \mathbb{R} \mid \beta_L(F) \geqslant xS_L(F) \text{ for any } F \in \mathcal{D}_L^{\mathrm{ud}}\}. \tag{6.60}$$

By definition, one obtains

$$C_L \geqslant \zeta^{\mathrm{ud}}(L). \tag{6.61}$$

Lemma 6.4. For any $L \in \mathrm{Amp}(X)$, we have

$$\zeta(L) = \zeta^{\mathrm{ud}}(L). \tag{6.62}$$

Proof. By definitions of $\zeta^{\mathrm{ud}}(L)$ and $\zeta(L)$, we have

$$\zeta(L) \leqslant \zeta^{\mathrm{ud}}(L). \tag{6.63}$$

For any $F \notin \mathcal{D}_L^{\mathrm{ud}}$, then

$$\beta_L(F) \geqslant C_L S_L(F) \geqslant \zeta^{\mathrm{ud}}(L) S_L(F). \tag{6.64}$$

Thus, for any $F \in \mathrm{PDiv}_{/X}$, we have

$$\beta_L(F) \geqslant \zeta^{\mathrm{ud}}(L) S_L(F), \tag{6.65}$$

i.e.

$$\zeta(L) \geqslant \zeta^{\mathrm{ud}}(L). \tag{6.66}$$

∎

Theorem 6.2 ([40][Theorem 13]). The uniformly valuative stability threshold

$$\mathrm{Amp}(X) \ni L \mapsto \zeta(L) \in \mathbb{R} \tag{6.67}$$

is continuous on the ample cone.

Proof. For any $L \in \mathrm{Amp}(X)$ and any $\varepsilon > 0$, we aim to show that there exists a small open neighbourhod U_θ of L in $\mathrm{Amp}(X)$ such that for any $L' \in U_\theta$ satisfying

$$|\zeta(L') - \zeta(L)| < \varepsilon. \tag{6.68}$$

By Theorem 6.1, for any $L' \in U_\theta$ satisfies the following inequality

$$\beta_{L'}(F) - \beta_L(F) \geqslant -f(\theta) S_{L'}(F) \tag{6.69}$$

for any $F \in \mathcal{D}_{L'}^{\mathrm{ud}}$, where f is a continuous function with $f(\theta) \to 0$ as $\theta \to 0$. Moreover, f only depends on X and L. Thus, we have

$$\begin{aligned}\beta_{L'}(F) \geqslant& \zeta(L) S_L(F) - f(\theta) S_{L'}(F)\\ =&(\zeta(L) + c_L) S_L(F) - c_L S_L(F) - f(\theta) S_{L'}(F)\end{aligned}$$

$$\geqslant \left((\zeta(L)+c_L)s^-(\theta)-c_Ls^+(\theta)-f(\theta)\right)S_{L'}(F)$$
$$=\left(\zeta(L)-(1-s^-(\theta))\zeta(L)-c_L(s^+(\theta)-s^-(\theta))-f(\theta)\right)S_{L'}(F) \tag{6.70}$$

for any $F\in\mathcal{D}^{\mathrm{ud}}_{L'}$. Thus one obtains

$$\zeta(L')=\zeta^{\mathrm{ud}}(L')\geqslant\zeta(L)-\left((1-s^-(\theta))\zeta(L)+c_L(s^+(\theta)-s^-(\theta))+f(\theta)\right). \tag{6.71}$$

We can take a small enough constant $\theta>0$ such that

$$(1-s^-(\theta))\zeta(L)+c_L(s^+(\theta)-s^-(\theta))+f(\theta)<\varepsilon. \tag{6.72}$$

Thus, we have

$$\zeta(L')-\zeta(L)>-\varepsilon. \tag{6.73}$$

On the other hand, by replacing L by L' and write $L=L'-\theta H$ in Theorem 6.1, we have

$$\beta_L(F)-\beta_{L'}(F)\geqslant -f(\theta)S_L(F) \tag{6.74}$$

for any $F\in\mathcal{D}^{\mathrm{ud}}_{L}$, where f is a continuous function with $f(\theta)\to 0$ as $\theta\to 0$. Moreover, f only depends on X and L. Similarly, we can compute

$$\beta_L(F)\geqslant\left(\zeta(L')s^+(\theta)^{-1}-c_{L'}(s^-(\theta)^{-1}-s^+(\theta)^{-1})-f(\theta)\right)S_L(F), \tag{6.75}$$

for any $F\in\mathcal{D}^{\mathrm{ud}}_{L}$. One obtains

$$\zeta(L)=\zeta^{\mathrm{ud}}(L)\geqslant\zeta(L')s^+(\theta)^{-1}-c_{L'}(s^-(\theta)^{-1}-s^+(\theta)^{-1})-f(\theta). \tag{6.76}$$

Then, we have

$$\zeta(L')\leqslant\zeta(L)+(s^+(\theta)-1)\zeta(L)+c_{L'}(s^+(\theta)s^-(\theta)^{-1}-1)+s^+(\theta)f(\theta). \tag{6.77}$$

One can choose a $c_{L'}$ depending on L' continuously since $\delta(\cdot)$, $\mu(\cdot)$ and $\tilde{s}(\cdot)$ are continuous on $\mathrm{Amp}(X)$. Then we take $\theta>0$ small enough such that

$$(s^+(\theta)-1)\zeta(L)+c_{L'}(s^+(\theta)s^-(\theta)^{-1}-1)+s^+(\theta)f(\theta)<\varepsilon. \tag{6.78}$$

Thus, we have

$$\zeta(L')-\zeta(L)<\varepsilon. \tag{6.79}$$

Together with (6.73), we finish the proof of Theorem 6.2. ∎

Chapter 7 Applications of valuative stability

7.1 Valuative stability for transcendental classes

In this section, let X be a projective manifold. We extend the valuative stability to the Kähler cone of projective manifolds.

Denote by $\mathcal{K}$ the *Kähler cone* of X and $\mathcal{E}$ the *pseudo-effective cone* in $H^{1,1}(X,\mathbb{R})$. The interior $\mathcal{E}^\circ$ of the psef cone is an open subcone, whose element is called *big*.

Recall the definition of the volume of a big class α in E° (see Section 3.2 or [72]Definition 3.2),

$$\mathrm{Vol}(\alpha) := \sup_{T\in\alpha}\int_{\widetilde{X}}\gamma^n > 0, \tag{7.1}$$

where the supremum is taken over all Kähler currents $T\in\alpha$ with logarithmic poles, and $\pi^*T = [E]+\gamma$ with respect to some modification $\pi : \widetilde{X}\to X$ for an effective $\mathbb{Q}$-divisor E and a closed semi-positive form γ (or see [75]Definition 1.17 for a definition in the sense of the pluripotential theory).

Let $\alpha\in K$ be a Kähler class of X, for any prime divisor F over X, then $\mathrm{Vol}(\alpha-x[F])$ is well-defined for some small $x>0$. Since $\pi^*\alpha$ may not be Kähler on Y, but it is still big. Therefore, by the openness of the big cone E°, we can define the pseudo-effective threshold of $F\in\mathrm{PDiv}_{/X}$ with respect to the Kähler class α as

$$\tau_\alpha(F) := \{x\in\mathbb{R} \mid \mathrm{Vol}(\alpha-x[F])>0\}. \tag{7.2}$$

It follows that the S-invariant is well-defined,

$$S_\alpha(F) := \int_0^\infty \mathrm{Vol}(\alpha-x[F])dx. \tag{7.3}$$

Similarly, for any Kähler class α, we also define

$$\mu(\alpha) := \frac{c_1(X)\cdot\alpha^{n-1}}{\alpha^n}, \tag{7.4}$$

$$s(\alpha) := \sup\{s\in\mathbb{R} \mid c_1(X)-s\alpha \text{ is Kähler }\}, \tag{7.5}$$

and

$$\tilde{s}(\alpha) := \inf\{s\in\mathbb{R} \mid -c_1(X)+s\alpha \text{ is Kähler }\}. \tag{7.6}$$

We also have $s(\alpha)\leqslant\mu(\alpha)\leqslant\tilde{s}(\alpha)$.

In [72], the authors established the perfect theory of the positive intersection product of big classes on compact Kähler manifolds (see Remark 3.1 for the definition).

Theorem 7.1 ([72] Theorem 3.5). Let X be a compact Kähler manifold. We denote here by $H^{k,k}_{\geqslant 0}(X)$ the cone of cohomology classes of type (k,k) which have non-negative intersection with all closed semi-positive smooth forms of bidegree $(n-k, n-k)$.

(i) For each integer $k = 1, 2, \cdots, n$, there exists a canonical "movable intersection product"

$$E \times \cdots \times E \to H^{k,k}_{\geqslant 0}(X), \quad (\alpha_1, \cdots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle \tag{7.7}$$

such that $\mathrm{Vol}(\alpha) = \langle \alpha^n \rangle$ whenever α is a big class (see Remark 3.1).

(ii) The product is increasing, homogeneous of degree 1 and super-additive in each argument, i.e.

$$\langle \alpha_1 \cdots (\alpha_j' + \alpha_j'') \cdots \alpha_k \rangle \geqslant \langle \alpha_1 \cdots \alpha_j' \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha_j'' \cdots \alpha_k \rangle. \tag{7.8}$$

It coincides with the ordinary intersection product when the $\alpha_j \in \overline{K}$ are nef classes.

(iii) The movable intersection product satisfies the Teissier-Hovanskii inequality

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_n \rangle \geqslant (\langle \alpha_1^n \rangle)^{1/n} \cdots (\langle \alpha_n^n \rangle)^{1/n}. \tag{7.9}$$

It follows that the β-invariant is well-defined for any Kähler class. For any $\alpha \in \mathcal{K}$, we define

$$\beta_\alpha(F) := A_X(F)\mathrm{Vol}(\alpha) + n\mu(\alpha) \int_0^{+\infty} \mathrm{Vol}(\alpha - x[F])dx - \int_0^{+\infty} n\langle (\alpha - x[F])^{n-1} \rangle \cdot c_1(X)dx. \tag{7.10}$$

Therefore, we can extend the valuative stability to any Kähler class,

Definition 7.1. For any $\alpha \in \mathcal{K}$, (X, α) is called

(i) *valuatively semistable* if

$$\beta_\alpha(F) \geqslant 0 \tag{7.11}$$

for any prime divisor F over X;

(ii) *valuatively stable* if

$$\beta_\alpha(F) > 0 \tag{7.12}$$

for any non-trivial prime divisor F over X;

(iii) *uniformly valuatively stable* if there exists an $\varepsilon_\alpha > 0$ such that

$$\beta_\alpha(F) \geqslant \varepsilon_\alpha S_\alpha(F) \tag{7.13}$$

for any prime divisor F over X.

The positive intersection product $\langle \alpha_1 \cdot \cdots \cdot \alpha_p \rangle$ depends continuously on the p-tuple

$(\alpha_1, \cdots, \alpha_p)$ for any big classes $\alpha_1, \cdots, \alpha_p$ (see the below in [75][Definition 1.17]).

If γ is psef and α is big, by ((ii)) and ((iii)) of Theorem 7.1, then we have

$$\mathrm{Vol}(\alpha + \gamma) \geqslant \mathrm{Vol}(\alpha). \tag{7.14}$$

A well-known result about the differentiability of the volume function on E°, due to D. Witt Nyström [97], is stated as follows,

Theorem 7.2 ([97][Theorem C]). On a projective manifold X the volume function is continuously differentiable on the big cone E° with

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}(\alpha + t\gamma) = n\langle \alpha^{n-1} \rangle \cdot \gamma. \tag{7.15}$$

for any $\alpha \in \mathcal{E}^\circ$ and any $\gamma \in H^{1,1}(X, \mathbb{R})$.

Therefore, we have the similar integration by part type formula:

$$\int_0^{+\infty} n\langle (\alpha - x[F])^{n-1} \rangle \cdot \alpha dx = (n+1) \int_0^{\infty} \mathrm{Vol}(\alpha - x[F]) dx, \tag{7.16}$$

for any $\alpha \in \mathcal{K}$ and any prime divisor F over X.

It follows that these proofs of Theorem 6.1 and Theorem 5.3 can also work for the Kähler cone. In other words, the openness of uniformly valuative stability also holds on the Kähler cone. We summary as follows,

Theorem 7.3 ([40][Theorem 3]). For a projective manifold X, the uniformly valuative stability locus

$$\widehat{\mathrm{UV}s} := \{\alpha \in \mathcal{K} \mid (X, \alpha) \text{ is uniformly valuatively stable}\} \tag{7.17}$$

is an open subcone of the Kähler cone $\mathcal{K}$.

7.2 Valuative J-stability

Let (X, L) be a polarized manifold and H be an ample line bundle on X. Fix the Kähler metrics $\chi \in c_1(H) =: \theta$ and $\omega \in c_1(L) =: \alpha$.

We consider the following J-equation

$$\mathrm{tr}_{\omega_\phi} \chi = c, \text{ i.e. } n\chi \wedge \omega_\phi^{n-1} = c\omega_\phi^n, \tag{7.18}$$

where

$$c = \frac{n \int_X \chi \wedge \omega^{n-1}}{\int_X \omega^n} = n\frac{H \cdot L^{n-1}}{L^n} =: n\mu_H(L). \tag{7.19}$$

It is well-known that the J-equation is the critical point of the $\mathcal{J}_\chi$ functional, defined as follows

$$\mathcal{J}_\chi(\phi) = E_\chi(\phi) - cE(\phi). \tag{7.20}$$

Indeed, for a smooth path $\phi_t \in \mathcal{H}(\omega)$, the derivative of $\mathcal{J}_\chi$ is

$$\frac{d}{dt}\mathcal{J}_\chi(\phi_t) = \int_X \frac{\partial \phi_t}{\partial t}\left(n\chi \wedge \omega_{\phi_t}^{n-1} - c\omega_{\phi_t}^n\right). \tag{7.21}$$

For each (normal, semiample, dominating) test configuration $(\mathcal{X}, \mathcal{L})$, the non-Archimedean $\mathcal{J}_H^{\mathrm{NA}}$ functional is defined by

$$\begin{aligned}\mathcal{J}_H^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) =& (E^H)^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) - cE^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \\ =& \frac{1}{\mathrm{Vol}(L)}(\rho^* H \cdot \overline{\mathcal{L}}^n) - n\mu_H(L)\frac{1}{(n+1)\mathrm{Vol}(L)}(\overline{\mathcal{L}}^{n+1})\end{aligned} \tag{7.22}$$

where $\rho : \overline{\mathcal{X}} \to X_{\mathbb{P}^1} \to X$.

Definition 7.2. We say that (X, L) is *uniform J^H-stable* (resp. *uniformly slope J^H-stable*) if there exists a constant $\varepsilon > 0$ such that

$$\mathcal{J}_H^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geqslant \varepsilon J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \tag{7.23}$$

for any test configuration (resp. deformation to the normal cone, see Section 7.2.1 for the definition) $(\mathcal{X}, \mathcal{L})$.

Recently, G. Chen [49] showed the existence of solutions of J-equation under a uniformly numerical condition.

Theorem 7.4 ([49][Theorem 1.1]). Let X be a compact Kähler manifold. Suppose ω, χ are Kähler metrics on X. Let $c > 0$ be a constant such that

$$\int_X n\chi \wedge \omega^{n-1} = c\int_X \omega^n. \tag{7.24}$$

The following statement are equivalent:

(i) There exists a unique smooth function ϕ up to a constant such that its Kähler form ω_ϕ satisfies the J-equation

$$\mathrm{tr}_{\omega_\phi}\chi = c. \tag{7.25}$$

(ii) There exists a unique smooth function ϕ up to a constant such that ϕ is the critical point of the $\mathcal{J}_\chi$ functional.

(iii) The $\mathcal{J}_\chi$ functional is *coercive*; in other words, there exists a constant $\varepsilon > 0$ and $C > 0$ such that

$$\mathcal{J}_\chi(\phi) \geqslant \varepsilon J(\phi) - C. \tag{7.26}$$

(iv) $(X, [\omega])$ is uniformly J-stable; in other words, there exists a constant $\varepsilon > 0$ such that

$$J_{[\chi]}(\mathcal{X},\Omega)\geqslant \varepsilon J_{[\omega]}(\mathcal{X},\Omega), \tag{7.27}$$

for all Kähler test configurations $(\mathcal{X},\Omega)$ (see [41][Definition 2.10]), where numerical invariants $J_{[\chi]}(\mathcal{X},\Omega)$ and $J_{[\omega]}(\mathcal{X},\Omega)$ (see [41][Definition 6.3]).

(v) $(X,[\omega])$ is uniformly slope J-stable; in other words, there exists a constant $\varepsilon>0$ such that

$$J_{[\chi]}(\mathcal{X},\Omega)\geqslant \varepsilon J_{[\omega]}(\mathcal{X},\Omega), \tag{7.28}$$

for any deformation to the normal cone $(\mathcal{X},\Omega)$ with respect to any analytic subvariety Z (see [41][Example 2.11 (ii)]).

(vi) There exists a constant $\varepsilon>0$ such that

$$\int_V (c-(n-p)\varepsilon)\omega^p - p\chi\wedge\omega^{p-1}\geqslant 0 \tag{7.29}$$

for all p-dimensional analytic subvarieties V with $p=1,2,\cdots,n$.

Remark 7.1. (i) When $[\omega]=c_1(L)$ and $[\chi]=c_1(H)$, then the numerical invariants $J_{[\chi]}(\mathcal{X},\Omega)$ and $J_{[\omega]}(\mathcal{X},\Omega)$ are nothing but $\mathcal{J}_H^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ and $J^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ resp.

(ii) When the Chen's paper [49] was under reviewed, Datar and Pingali [98] removed the technical ε-term in (7.29) in the projective case. Later, Song [99] solved this technical issue in the compact Kähler case. Then, this solves the Lejmi-Székelyhidi's original conjecture [45], which is that the solvability of the J-equation

$$\mathrm{tr}_{\omega_\phi}\chi=c \tag{7.30}$$

is equivalent to

$$\int_V c\omega^p - p\chi\wedge\omega^{p-1}>0 \tag{7.31}$$

for all p-dimensional analytic subvarieties V with $p=1,2,\cdots,n$.

(iii) The equivalence of (i) and (ii) follows from the formula (7.21). The equivalence of (i) and (iii) is due to [46]. (iii) implying (iv) is due to [41]. It is trivial that (iv) implies (v). By [45], (v) implies (vi). The main distribution of [49] is to show that (vi) implies (i).

For convenience, we consider the polarized case, namely, a smooth polarized variety (X,L) with a ample line bundle H. Motivated by Fujita-Li criterion for the Kähler-Einstein equation, we hope to study the uniform J^H-stability in terms of valuations.

For any prime divisor F over X, from the definition of β_L-invariant (5.14), we define the j_H-invariant as follows

$$j_H(F) := S_H(F) - cS(F), \tag{7.32}$$

where

$$S_H(F) = \frac{n}{\mathrm{Vol}(L)} \int_0^{+\infty} \langle (L - xF)^{n-1} \rangle . H dx \tag{7.33}$$

and

$$S(F) = \frac{1}{\mathrm{Vol}(L)} \int_0^{+\infty} \mathrm{Vol}(L - xF) dx. \tag{7.34}$$

Similar with $\delta(X)$-invariant in Section 5.1, we define the J^H-stability threshold as

$$\gamma_H(L) := \inf_{F \in \mathrm{PDiv}_{/X}} \frac{S_H(F)}{S(F)} > 0. \tag{7.35}$$

Definition 7.3. We say that (X, L) is

(i) *valuatively J^H-semistable* if $j_H(F) \geqslant 0$ for any prime divisor F over X;

(ii) *valuatively J^H-stable* if $j_H(F) > 0$ for any non-trivial prime divisor F;

(iii) *uniformly valuatively J^H-stable* if there exists a constant $\varepsilon > 0$ such that

$$j_H(F) \geqslant \varepsilon j(F) \tag{7.36}$$

for any prime divisor F over X.

Obviously, uniformly valuative J^H-stable is equivalent to $\gamma_H(L) > c = n\mu_H(L)$.

For any prime divisor F over X, it induces a $\mathbb{Z}$-filtration $\mathscr{F}$ on the section ring R, given by

$$\mathscr{F}^\lambda R_k = \begin{cases} H^0(X, kL - \lambda F) & \text{if } \lambda \geqslant 0, \\ H^0(X, L^k) & \text{if } \lambda < 0. \end{cases} \tag{7.37}$$

Note that the filtration $\mathscr{F}$ is finite generated when F is a dreamy divisor, which corresponds to a test configuration with integral central fiber.

Then $\mathscr{F}$ induces a $\mathcal{I}$-model test curve ψ_τ, i.e., $\psi_r = P[\psi_r]_\mathcal{I}$ as in Section 4.6, given by

$$\psi_r = \sup_{k \in \mathbb{Z}_+}{}^{*} \left(\frac{1}{k} \sup{}^{*} \left\{ \log |s|^2_{h^k} \;\middle|\; s \in \mathcal{F}^{kr} R_k, \sup |s|_{h^k} \leqslant 1 \right\} \right), \tag{7.38}$$

where " $*$ " is the upper semicontinuous regularization and h is a smooth metric on L such that $c_1(h) = \omega$. It is easy to see that $r^+ = \tau_L(F)$ by the definition of r^+. Moreover, $\psi_r \equiv 0$ for $r \leqslant 0$.

Claim 7.5. For $0 \leqslant r < r^+$, we have

$$\nu(\psi_r, F) := \inf_{z\in F} \nu(\psi_r, z) \geqslant \tau, \tag{7.39}$$

where $\nu(\psi_r, z)$ is the Lelong number of the ω-psh function ψ_r at z.

Proof. It is a standard argument by Fekete Lemma. We follow the argument in [90]. For $k \in \mathbb{Z}_{>0}$, we set

$$\psi_r^k := \overset{*}{\sup}\{\log|s|^2_{h^k} \mid s \in \mathcal{F}^{k\tau} R_k, |s|^2_{h^k} \leqslant 1\} \leqslant 0. \tag{7.40}$$

By the multiplicativity of the filtration, we have

$$\psi_r^{k+m} \geqslant \psi_r^k + \psi_r^m \tag{7.41}$$

for all $k, m \in \mathbb{Z}_{>0}$. By Fekete's Lemma, then ψ_r is the upper semi-continuous regularization of the increasing limit ψ_r^k/k. By the monotonicity and the upper semi-continuity of Lelong numbers (see [100]), we conclude

$$\nu(\psi_r, F) = \inf_{k\in\mathbb{Z}_{>0}} \frac{1}{k} \inf\{\nu(\log|s|^2_{h^k}, F) \mid s \in \mathscr{F}^{kr} R_k\}. \tag{7.42}$$

Since $\nu(\log|s|^2_{h^k}, F) \geqslant r$, then we obtain $\nu(\psi_r, F) \geqslant r$. ∎

Since ψ_τ is $\mathcal{I}$-model, by Theorem 4.7, we have

$$\begin{aligned}\int_X \omega^n_{\psi_r} &= \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathcal{I}(k\psi_r)) \\ &\geqslant \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, \mathcal{F}^{kr} R_k) = \mathrm{Vol}(L - rF),\end{aligned} \tag{7.43}$$

since every element in $\mathcal{F}^{kr} R_k$ is obviously square integrable with respect to $k\psi_r$.

Lemma 7.1. For any prime divisor F over X and $\phi \in \mathrm{Psh}(X, \omega)$ satisfying

$$\nu(\phi, F) \geqslant x. \tag{7.44}$$

Then we have

$$\mathrm{Vol}(L - xF) \geqslant \int_X \omega^n_\phi \tag{7.45}$$

and

$$\langle (L - xF)^{n-1} \rangle . H \geqslant \int_X \omega^{n-1}_\phi \wedge \chi. \tag{7.46}$$

Proof. Fix a smooth hermitian metric h on $\mathcal{O}_Y(F)$, let s_F be the defining section of $\mathcal{O}_Y(F)$. Since $\nu(\varphi, F) \geqslant x$, then there exist $C > 0$ such that

$$\pi^*\phi \leqslant x \log|s_F|^2_h + C. \tag{7.47}$$

We set

$$\omega_h := -dd^c \log h \in c_1(\mathcal{O}_Y(F)). \tag{7.48}$$

Then one has

$$\pi^*\omega - x\omega_h \in [\pi^*\omega] - xc_1(F). \tag{7.49}$$

We consider

$$u := \pi^*\phi - x \log |s_F|_h^2. \tag{7.50}$$

Recall the Poincaré-Lelong formula

$$\mathrm{dd^c} \log |s_F|_h^2 = [F] + \mathrm{dd^c} \log h, \tag{7.51}$$

where $[F]$ is the integration current of F. Then one obtains

$$\mathrm{dd^c} u = \pi^*(\mathrm{dd^c}\phi) + x\omega_h, \text{ on } Y\backslash F. \tag{7.52}$$

It follows that

$$\pi^*\omega - x\omega_h + \mathrm{dd^c} u = \pi^*(\omega + \mathrm{dd^c}\phi), \text{ on } Y\backslash F. \tag{7.53}$$

Since u is quasi-psh and bounded from above on $Y\backslash F$, then u can be extended to a quasi-psh function on Y, denoted by u'. Since

$$\pi^*\omega - x\omega_h + \mathrm{dd^c} u' \in [\pi^*\omega] - xc_1(F), \tag{7.54}$$

by [75][Proposition 1.20], we have

$$\langle(\pi^*\alpha - x[F])^p\rangle \geqslant [\langle(\pi^*\omega - x\omega_h + \mathrm{dd^c} u')^p\rangle] \tag{7.55}$$

in $H^{p,p}(X, \mathbb{R})$, where $\langle\cdot\rangle$ is the movable intersection and " $\geqslant$ " means that the difference is pseudo-effective, i.e. having the non-negative intersection with all closed smooth semi-positive form. Thus, we have

$$\begin{aligned}
\mathrm{Vol}(\pi^*L - xF) =& \langle(\pi^*L - xF)^n\rangle \\
=& \langle(\pi^*\alpha - x[F])^n\rangle \\
\geqslant& \int_Y (\pi^*\omega - x\omega_h + \mathrm{dd^c} u')^n \\
=& \int_{Y\backslash F} (\pi^*\omega - x\omega_h + \mathrm{dd^c} u)^n \\
=& \int_{Y\backslash F} \pi^*(\omega + \mathrm{dd^c}\phi)^n \\
=& \int_X \omega_\phi^n,
\end{aligned} \tag{7.56}$$

where we used (7.55) for $p = n$. By nefness of $\pi^* H$ and (7.55) for $p = n-1$, we obtain

$$\begin{aligned}\langle (L - xF)^{n-1}\rangle . H =& \langle (\pi^*\alpha - x[F])^{n-1}\rangle \cdot \pi^*\theta \\ \geqslant& \left[\langle (\pi^*\omega - x\omega_h + dd^c u')^{n-1}\rangle\right] \cdot \pi^*\theta \\ =& \int_Y (\pi^*\omega - x\omega_h + dd^c u')^{n-1} \wedge \pi^*\chi \\ =& \int_{Y\setminus F} (\pi^*\omega - x\omega_h + dd^c u)^{n-1} \wedge \pi^*\chi \\ =& \int_{Y\setminus F} \pi^*(\omega + dd^c\phi)^{n-1} \wedge \pi^*\chi \\ =& \int_X \omega_\phi^{n-1} \wedge \chi. \end{aligned} \tag{7.57}$$

We finish the proof. ∎

Note that (7.46) also holds for the nef line bundle H.

By (7.43), (7.45), and Claim 7.5, we have

$$\int_X \omega_{\psi_r}^n = \mathrm{Vol}(L - rF) \tag{7.58}$$

for $0 \leqslant r < r^+$.

By (7.58), we can compute the Monge-Ampère energy of the test curve $\psi_\bullet$.

$$\begin{aligned}\mathbf{E}(\psi) =& r^+ + \frac{1}{V}\int_{-\infty}^{r^+} \left(\int_X \omega_{\psi_r}^n - \int_X \omega^n\right) dr \\ =& r^+ + \int_0^{r^+} \frac{1}{V}\mathrm{Vol}(L - rF) - 1 dr \\ =& \frac{1}{V}\int_0^{r^+} \mathrm{Vol}(L - rF) dr \\ =& S(F). \end{aligned} \tag{7.59}$$

By (7.46), we also obtain

$$\begin{aligned}\mathbf{E}_\chi(\psi) =& r^+ \frac{1}{V}\int_X \omega^{n-1}\wedge\chi + \frac{1}{V}\int_{-\infty}^{r^+}\left(\int_X \omega_{\psi_r}^{n-1}\wedge\chi - \int_X \omega^{n-1}\wedge\chi\right) dr \\ =& r^+\mu_H + \int_{-\infty}^{r^+}\left(\frac{1}{V}\int_X \omega_{\psi_r}^{n-1}\wedge\chi - \mu_H\right) dr \\ \leqslant& r^+\mu_H + \int_0^{r^+}\left(\frac{1}{V}\langle (L - rF)^{n-1}\rangle . H - \mu_H\right) dr \\ =& \frac{1}{V}\int_0^{r^+} \langle (L - rF)^{n-1}\rangle . H dr \\ =& S_H(F). \end{aligned} \tag{7.60}$$

Recall the inverse Legendre transform,

$$\phi_t := \sup_{r\in\mathbb{R}}(\psi_r + tr), \quad t \geqslant 0. \tag{7.61}$$

Since $\psi_\bullet$ is the $\mathcal{I}$-model test curve, by Theorem 4.9, we know that ϕ_t is a maximal geodesic ray. Moreover, by Theorem 4.8 and Theorem 4.10, we have

$$E'^{\infty}(\phi) = \mathbf{E}(\psi) \tag{7.62}$$

and

$$(E_\chi)'^{\infty}(\phi) = \mathbf{E}_\chi(\psi). \tag{7.63}$$

Then we have

$$\begin{aligned} j_H(F) =& S_H(F) - cS(F) \\ \geqslant& \mathbf{E}_\chi(\psi) - c\mathbf{E}(\psi) \\ =& (E_\chi)'^{\infty}(\phi) - cE'^{\infty}(\phi) \end{aligned} \tag{7.64}$$

By Theorem 4.5, we denote φ by the corresponding non-Archimedean finite energy functional of the maximal geodesic ray ϕ. By Theorem 4.5 and Theorem 4.6, then we have

$$E'^{\infty}(\phi) = E^{\mathrm{NA}}(\varphi), \quad \text{and} \quad (E_\chi)'^{\infty}(\Phi) = (E^H)^{\mathrm{NA}}(\varphi). \tag{7.65}$$

Thus, we obtain

$$j_H(F) \geqslant (E^H)^{\mathrm{NA}}(\varphi) - cE^{\mathrm{NA}}(\varphi) = \mathcal{J}_H^{\mathrm{NA}}(\varphi). \tag{7.66}$$

By Proposition 4.2 and 4.3, there exists a sequence φ_k in $\mathcal{H}^{\mathrm{NA}}(L)$ strongly converging to φ such that

$$\mathcal{J}_H^{\mathrm{NA}}(\varphi) = \lim_{k\to\infty} \mathcal{J}_H^{\mathrm{NA}}(\varphi_k), \quad \text{and} \quad J^{\mathrm{NA}}(\varphi) = \lim_{k\to\infty} J^{\mathrm{NA}}(\varphi_k). \tag{7.67}$$

On the other hand, we have

$$\begin{aligned} J^{\mathrm{NA}}(\varphi) =& \Lambda^{\mathrm{NA}}(\varphi) - E^{\mathrm{NA}}(\varphi) \\ =& \sup_{X^{\mathrm{an}}} \varphi - \mathbf{E}(\psi_\bullet) \\ =& \tau_L(F) - S(F) = j(F). \end{aligned} \tag{7.68}$$

If we assume that (X, L) is uniformly $\boldsymbol{J}^H$-stable, together with the above argument, we conclude

$$\begin{aligned} j_H(F) \geqslant& \lim_{k\to\infty} \mathcal{J}_H^{\mathrm{NA}}(\varphi_k) \\ \geqslant& \varepsilon \lim_{k\to\infty} \boldsymbol{J}^{\mathrm{NA}}(\varphi_k) = \varepsilon \boldsymbol{J}^{\mathrm{NA}}(\varphi) = \varepsilon j(F). \end{aligned} \tag{7.69}$$

Thus, we proved

Proposition 7.1. Uniform J^H-stability implies uniformly valuative J^H-stability. In other word, if the polarized manifold (X, L) has a unique solution of the J-equation (7.18), then

$$\gamma_H(L) > c. \tag{7.70}$$

7.2.1 Deformation to the normal cone

In this subsection, we compute the functional $\mathcal{J}_H^{\mathrm{NA}}$ for a special class of examples of test configurations, the so-called *deformation to the normal cone*.

Let Z be an irreducible subvariety of X. Set

$$\mathcal{X} = \mathrm{Bl}_{Z\times\{0\}}(X_{\mathbb{A}^1}) \quad \text{and} \quad \mathcal{L} = \rho^* L - P, \tag{7.71}$$

where P is the exceptional divisor. We have the following diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"] \arrow[rr, bend left, "\rho"] \arrow[dr, "p"'] & X_{\mathbb{P}^1} \arrow[r, "p_1"] \arrow[d, "p_2"] & X \\ & \mathbb{P}^1 & \end{tikzcd} \tag{7.72}$$

We may assume that $\mathcal{L}$ is ample. $(\mathcal{X}, \mathcal{L})$ is called the *deformation to the normal cone* with respect to the subvariety Z. Let $\widehat{X}$ be the strict transform of blow-up of X along Z with the exceptional divisor E. Then

$$\mathcal{X}_0 = \widehat{X} \cup_E P. \tag{7.73}$$

Set

$$L_s := L + sH, \text{ and } \mathcal{L}_s := \overline{\mathcal{L}} + s\rho^* H. \tag{7.74}$$

Take $k_s \in \mathbb{Z}_{>0}$ such that $k_s L_s$ and $k_s \mathcal{L}_s$ are line bundles. In particular, $k_0 = 1$. One has

$$\begin{aligned}(E^H)^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{\mathrm{Vol}(L)}(\rho^* H \cdot \overline{\mathcal{L}}^n) \\ &= \frac{1}{\mathrm{Vol}(L)} \frac{d}{ds}\bigg|_{s=0} \frac{(\overline{\mathcal{L}} + s\rho^* H)^{n+1}}{n+1} \\ &= \frac{1}{\mathrm{Vol}(L)} \frac{d}{ds}\bigg|_{s=0} \frac{\mathrm{Vol}(k_s L_s)}{(n+1)k_s^{n+1}} \frac{(k_s \mathcal{L}_s)^{n+1}}{\mathrm{Vol}(k_s L_s)} \\ &= \frac{1}{\mathrm{Vol}(L)} \frac{d}{ds}\bigg|_{s=0} \frac{1}{k_s}\mathrm{Vol}(L_s) E^{\mathrm{NA}}(\mathcal{X}, k_s \mathcal{L}_s)\end{aligned} \tag{7.75}$$

For $s \in \mathbb{R}$ small enough, $(\mathcal{X}, k_s \mathcal{L}_s)$ is a ample test configuration of $(X, k_s L_s)$.

Set

$$R_s := \bigoplus_{m=0}^{\infty} R_{s,m} := \bigoplus_{m=0}^{\infty} H^0(X, mk_s L_s). \tag{7.76}$$

We write

$$v_P = r(\mathrm{ord}_P) = \mathrm{ord}_E. \tag{7.77}$$

By Theorem 4.3, the associated filtration of the test configuration $(\mathcal{X}, k_s\mathcal{L}_s)$ is defined by

$$\begin{aligned}\mathscr{F}^{\lambda}_{(\mathcal{X},k_s\mathcal{L}_s)} R_{s,m} :=& \{f \in R_{s,m} \mid t^{\lfloor -\lambda \rfloor}\rho^* f \in H^0(\mathcal{X}, mk_s\mathcal{L}_s)\} \\ =& \{f \in R_{s,m} \mid v_P(f) + mb_P^{-1}\mathrm{ord}_P(-k_sP) \geqslant \lambda, v_{\mathrm{triv}}(f) + 0 \geqslant \lambda\} \\ =& \{f \in R_{s,m} \mid v_P(f) \geqslant mk_s + \lambda, \lambda \leqslant 0\} \\ =& \begin{cases} H^0(X, mk_sL_s - (mk_s+\lambda)E) & \text{if } \lambda \leqslant 0, \\ 0 & \text{if } \lambda > 0.\end{cases}\end{aligned} \tag{7.78}$$

We have

$$\lambda_{\min} = -k_s, \quad \text{and} \quad \lambda_{\max} = 0. \tag{7.79}$$

When $s = 0$, we obtain the associated filtration of $(\mathcal{X}, \mathcal{L})$

$$\mathscr{F}^{\lambda}_{(\mathcal{X},\mathcal{L})} R_m = \begin{cases} H^0(X, L - (1+\lambda)E) & \text{if } \lambda \leqslant 0, \\ 0 & \text{if } \lambda > 0.\end{cases} \tag{7.80}$$

By theorem 4.2, then the Duistermaat-Heckman measure of $(\mathcal{X}, k_s\mathcal{L}_s)$ is given by

$$\nu_s = -\frac{1}{\mathrm{Vol}(k_sL_s)}\frac{d}{dr}\mathrm{Vol}(R_s^{(r)}), \tag{7.81}$$

where

$$\mathrm{Vol}(R_s^{(r)}) := \lim_{m\to\infty}\frac{n!}{m^n}\dim \mathscr{F}^{mr}_{(\mathcal{X},k_s\mathcal{L}_s)} R_{s,r}. \tag{7.82}$$

By Lemma 4.1, we have

$$\begin{aligned} E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) =& \int_{-\infty}^{\infty} r d\nu(r) \\ =& \int_{-\infty}^{0} \left(\frac{1}{\mathrm{Vol}(L)}\mathrm{Vol}(R^{(r)}) - 1\right) dr \\ =& \int_{-1}^{0} \frac{1}{\mathrm{Vol}(L)}\mathrm{Vol}(L - (1+r)E)dr - 1 \\ =& \frac{1}{\mathrm{Vol}(L)}\int_0^1 \mathrm{Vol}(L - xE)dx - 1. \end{aligned} \tag{7.83}$$

Similarly, we obtain

$$\begin{aligned}E^{\mathrm{NA}}(\mathcal{X}, k_s\mathcal{L}_s) &= \int_{-\infty}^{\infty} r d\nu_s(r) \\ &= \int_{-\infty}^{0} \left(\frac{1}{\mathrm{Vol}(k_sL_s)}\mathrm{Vol}(R_s^{(r)}) - 1\right) dr \\ &= \int_{-k_s}^{0} \frac{1}{\mathrm{Vol}(k_sL_s)}\mathrm{Vol}(k_sL_s - (k_s + r)E)dr - k_s \\ &= \frac{1}{\mathrm{Vol}(L_s)} \int_{-k_s}^{0} \mathrm{Vol}(L_s - (1 + r/k_s)E)dr - k_s. \end{aligned} \tag{7.84}$$

Then, by (7.75), one has

$$\begin{aligned}(E^H)^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{\mathrm{Vol}(L)} \frac{d}{ds}\bigg|_{s=0} \left(\int_{-k_s}^{0} \mathrm{Vol}(L_s - (1 + r/k_s)E)\frac{dr}{k_s} - \mathrm{Vol}(L_s)\right) \\ &= \frac{1}{\mathrm{Vol}(L)} \frac{d}{ds}\bigg|_{s=0} \left(\int_{0}^{1} \mathrm{Vol}(L + sH - xE)dx - \mathrm{Vol}(L_s)\right) \\ &= \frac{n}{\mathrm{Vol}(L)} \int_0^1 \langle (L - xE)^{n-1}\rangle \cdot H dx - n\mu_H. \end{aligned} \tag{7.85}$$

It follows that

$$\begin{aligned}\mathcal{J}_H^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{\mathrm{Vol}(L)} \left(\rho^* H \cdot \overline{\mathcal{L}}^n\right) - n\mu_H \frac{1}{(n+1)\mathrm{Vol}(L)} \left(\overline{\mathcal{L}}^{n+1}\right) \\ &= (E^H)^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) - cE^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \\ &= \frac{n}{\mathrm{Vol}(L)} \int_0^1 \langle (L - xE)^{n-1}\rangle \cdot H dx - c\frac{1}{\mathrm{Vol}(L)} \int_0^1 \mathrm{Vol}(L - xE)dx. \end{aligned} \tag{7.86}$$

Proposition 7.2. Let (X, L) be a polarized variety and $(\mathcal{X}, \mathcal{L})$ be the deformation to the normal cone with respect to a irreducible subvariety $Z \subset X$. As above notation, then we have

$$\mathcal{J}_H^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{n}{\mathrm{Vol}(L)} \int_0^1 \langle (L - xE)^{n-1}\rangle \cdot H dx - c\frac{1}{\mathrm{Vol}(L)} \int_0^1 \mathrm{Vol}(L - xE)dx, \tag{7.87}$$

where H is a ample line bundle on X and

$$c = n\frac{H \cdot L^{n-1}}{L^n}. \tag{7.88}$$

Unfortunately, this can not imply slope J^H-stability since we can not deal with the prime divisors F over X with $\tau_L(F) > 1$.

7.3 Upper bound of the volume

An interesting application of valuative criterion of Fano manifolds is to obtain the volume upper bound of Fano manifolds, due to [51].

In this section, as applications of valuative stability, we consider the volume upper bound of polarized varieties. In particular, we obtain an upper bound of the volume for K-semistable toric variety.

Proposition 7.3. Let (X, L) be a polarized variety of dimension n. Assume (X, L) is valuatively semistable. Then we have

(i) if $\mu(L) > 0$, then

$$\mu(L)^n \mathrm{Vol}(L) \leqslant (n+1)^n \left(1 + (\mu(L) - s(L))\tau_L(F)\right)^n ; \tag{7.89}$$

(ii) if $\mu(L) < 0$, then

$$(-\mu(L))^n \mathrm{Vol}(L) \leqslant \left(1 + \frac{1}{n}\right)^n \left(1 - s(L)\tau_L(F)\right)^n , \tag{7.90}$$

where F is an exceptional divisor of blow-up at a smooth point.

Remark 7.2. By definition of $\tau_{kL}(F)$, it is easy to see that

$$\tau_{kL}(F) = k\tau_L(F). \tag{7.91}$$

From this, we note that inequality (7.89) and (7.90) are scaling invariant under the multiple of L. Thus, we can assume a normalization condition of L as follows,

$$\mu(L) = \begin{cases} 1 & \text{if } \mu(L) > 0, \\ -1 & \text{if } \mu(L) < 0. \end{cases} \tag{7.92}$$

Under the normalization condition, then the upper bound of volume in Proposition 7.3 becomes

$$\mathrm{Vol}(L) \leqslant (n+1)^n \left(1 + (1 - s(L))\tau_L(F)\right)^n \tag{7.93}$$

and

$$\mathrm{Vol}(L) \leqslant \left(1 + \frac{1}{n}\right)^n \left(1 - s(L)\tau_L(F)\right)^n . \tag{7.94}$$

In [51], the author gave an interesting lower bound of the volume of $L - xF$.

Lemma 7.2 ([51]Theorem 2.3). Let X be an n-dimensional projective variety with $n \geqslant 2$, let L be an ample $\mathbb{Q}$-divisor on X, $p \in X$ be a smooth closed point, $\nu : \widehat{X} \to X$ be the blowup at p, and $F \subset \widehat{X}$ be the exceptional divisor of ν. Then for any $x \in \mathbb{R}_{\geqslant 0}$, we have

$$\mathrm{Vol}(L - xF) \geqslant ((L - xF)^n) = (L^n) - x^n. \tag{7.95}$$

Let F be the exceptional divisor of blowup at a smooth closed point, by Lemma 5.1 and (7.95), we have

$$\tau_L(F) \geqslant \frac{n+1}{n} \frac{1}{\mathrm{Vol}(L)} \int_0^\infty \mathrm{Vol}(L - xF) dx$$

$$\begin{aligned}&\geqslant \frac{n+1}{n}\frac{1}{\mathrm{Vol}(L)}\int_0^{\tau_L(F)}(\mathrm{Vol}(L)-x^n)dx\\&\geqslant \frac{n+1}{n}\left(\tau_L(F)-\frac{1}{\mathrm{Vol}(L)}\int_0^{\tau_L(F)}x^n dx\right).\end{aligned} \tag{7.96}$$

Thus, one has

$$\frac{1}{(n+1)\mathrm{Vol}(L)}\tau_L(F)^{n+1}\geqslant \frac{1}{n+1}\tau_L(F). \tag{7.97}$$

Then we obtain

$$\tau_L(F)\geqslant \sqrt[n]{\mathrm{Vol}(L)}. \tag{7.98}$$

It follows from (7.95) and (7.98) that

$$\begin{aligned}\int_0^{\infty}\mathrm{Vol}(L-xF)dx\geqslant&\int_0^{\sqrt[n]{\mathrm{Vol}(L)}}(\mathrm{Vol}(L)-x^n)dx\\=&\sqrt[n]{\mathrm{Vol}(L)}\frac{n}{n+1}\mathrm{Vol}(L).\end{aligned} \tag{7.99}$$

Proof of Proposition 7.3 Let F be the exceptional divisor of blowup at a smooth closed point. By the construction of blowup at a smooth closed point, then it is easy to compute

$$A_X(F)=n. \tag{7.100}$$

(i) By (5.29), we have

$$\begin{aligned}\beta_L(F)=&A_X(F)\mathrm{Vol}(L)+(n\mu(L)-(n+1)s(L))S_L(F)\\&-\int_0^{+\infty}\mathrm{Vol}'(L-xF).(-s(L)L-K_X)dx\\=&n\mathrm{Vol}(L)+(n+1)(\mu(L)-s(L))S_L(F)\\&-\mu(L)S_L(F)-\int_0^{+\infty}\mathrm{Vol}'(L-xF).(-s(L)L-K_X)dx.\end{aligned} \tag{7.101}$$

By the assumption, we obtain

$$\begin{aligned}&n\mathrm{Vol}(L)+(n+1)(\mu(L)-s(L))S_L(F)\\\geqslant&\mu(L)S_L(F)+\int_0^{+\infty}\mathrm{Vol}'(L-xF).(-s(L)L-K_X)dx\\\geqslant&\mu(L)S_L(F),\end{aligned} \tag{7.102}$$

where we have used the definition of $s(L)$ for the second inequality.

By (7.99) and $\mu(L)\geqslant 0$, we have

$$n\mathrm{Vol}(L)+(n+1)(\mu(L)-s(L))S_L(F)\geqslant \mu(L)\sqrt[n]{\mathrm{Vol}(L)}\frac{n}{n+1}\mathrm{Vol}(L). \tag{7.103}$$

Thus, one has

$$\mu(L)\sqrt[n]{\operatorname{Vol}(L)} \leqslant (n+1)\left(1+(\mu(L)-s(L))\frac{S_L(F)}{\operatorname{Vol}(L)}\frac{n+1}{n}\right). \tag{7.104}$$

By Lemma 5.1, we obtain

$$\mu(L)\sqrt[n]{\operatorname{Vol}(L)} \leqslant (n+1)(1+(\mu(L)-s(L))\tau_L(F)). \tag{7.105}$$

(ii) Since $\beta_L \geqslant 0$, by (5.29), we have

$$\begin{aligned}&A_X(F)\operatorname{Vol}(L)-(n+1)s(L)S_L(F)\\ \geqslant&(-n\mu(L))S_L(F)+\int_0^{+\infty}\operatorname{Vol}'(L-xF).(-s(L)L-K_X)dx\\ \geqslant&(-n\mu(L))S_L(F).\end{aligned} \tag{7.106}$$

By (7.99), $A_X(F)=n$ and $\mu(L)\leqslant 0$, we obtain

$$\operatorname{Vol}(L)-(1+\frac{1}{n})s(L)S_L(F) \geqslant (-\mu(L))\sqrt[n]{\operatorname{Vol}(L)}\frac{n}{n+1}\operatorname{Vol}(L). \tag{7.107}$$

By Lemma 5.1, we have

$$(-\mu(L))\sqrt[n]{\operatorname{Vol}(L)} \leqslant (1+\frac{1}{n})(1-s(L)\tau_L(F)) \tag{7.108}$$

∎

The upper bound in Proposition 7.3 is a coarse estimate. We will give a refined upper bound for toric varieties later.

7.3.1 Upper bound of the volume of polarized toric varieties

The toric variety is a special class of examples to study K-stability and valuative stability. In this subsection, we give a more precise upper bound of the volume on toric variety. We follow the notation of [101].

Let $X = X_\Sigma$ be a n-dimensional projective normal toric variety associated to a fan $\Sigma \subset N_\mathbb{R}$, where N is the lattice of all one-parameter subgroups of the torus $T_N \simeq (\mathbb{C}^*)^n$ and the fan Σ in $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ is a collection of cones such that each cone σ in Σ is generated by finite many elements in N and $\sigma \cap \sigma' \in \Sigma$ for any two cones $\sigma, \sigma' \in \Sigma$. We denote by $M = N^*$ the lattice of characters of T_N and $\Sigma(1)$ the set of rays of the fan Σ. Each ray $\rho \in \Sigma(1)$ determines a prime divisor D_ρ and an element $u_\rho \in N$, namely the (unique) primitive vector in $\rho \cap N$. The ample line bundle corresponds to a full dimensional lattice polytope P (uniquely determined by the linear equivalence of line bundle up to a translation) whose fan is Σ.

Let $L = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a ample divisor. The lattice polytope P_L associated to L

is

$$P_L = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geqslant -a_\rho \text{ for all } \rho \in \Sigma(1)\}. \tag{7.109}$$

We improve the upper bound in Proposition 7.3 for the toric varieties.

Theorem 7.6. Let (X, L) be a polarized toric variety of dimension n. Assume (X, L) is K-semistable. Then we have

$$\sqrt[n]{\mathrm{Vol}(L)} \leqslant \max_\rho a_\rho \left(1 + \frac{n}{n+1}\mu(L)\tau_L(F)\right) \tag{7.110}$$

where F is an exceptional divisor of blowup at a smooth point.

Remark 7.3. (i) As in Remark 7.2, the upper bound (7.110) becomes

$$\sqrt[n]{\mathrm{Vol}(L)} \leqslant \max_\rho a_\rho \left(1 + \frac{n}{n+1}\tau_L(F)\right) \tag{7.111}$$

under the normalization condition $\mu(L) = 1$.

(ii) For toric varieties, we have an explicit formula for $\tau_L(F)$. Let $\sigma = \mathrm{Cone}(u_1, \cdots, u_n)$ be a smooth cone in Σ such that $\{u_i\}_{i=1}^n$ is a basis of N. We denote $u_0 := u_1 + \cdots + u_n$. Then u_0 corresponds to the valuation ord_F. Due to Blum-Jonsson [95][Corollary 7.7], one has

$$\tau(F) = \max_{v \in \mathrm{Vert}(P)} \langle v, u_0 \rangle - \psi(u_0) = \max_{v \in \mathrm{Vert}(P)} \langle v, u_0 \rangle - \min_{v \in \mathrm{Vert}(P)} \langle v, u_0 \rangle, \tag{7.112}$$

where ψ is the support function of L and $\mathrm{Vert}(P)$ denotes the set of vertices of P.

Proof of Theorem 7.6 It is well known that K-semistability of (X, L) implies the original Futaki invariant of (X, L) vanishes. By [34][Theorem 1.2], then (X, L) is valuative semistable, i.e., for any toric prime divisor F over X, we have

$$\beta_L(F) := A_X(F)\mathrm{Vol}(L) + n\mu(L) \int_0^{+\infty} \mathrm{Vol}(L - xF) dx + \int_0^{+\infty} \mathrm{Vol}'(L - xF).K_X dx \geqslant 0. \tag{7.113}$$

We take F as the exceptional divisor of blowup at a smooth point, induced by $u_0 := u_1 + \cdots + u_n$. One has

$$A_X(F) = 1 + \cdots + 1 = n. \tag{7.114}$$

Writing $L = \sum_\rho a_\rho D_\rho$ and $K_X = -\sum_\rho D_\rho$. By (7.113), we have

$$\begin{aligned} n\mathrm{Vol}(L) + n\mu S_L(F) &\geqslant n \int_0^{+\infty} \langle (L - xF)^{n-1} \rangle .(-K_X) dx \\ &= n \int_0^{+\infty} \langle (L - xF)^{n-1} \rangle .(\sum_\rho D_\rho) dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{n}{\max_\rho a_\rho} \int_0^{+\infty} \langle (L - xF)^{n-1} \rangle . (\sum_\rho (\max_\rho a_\rho) D_\rho) dx \\
&\geqslant \frac{n}{\max_\rho a_\rho} \int_0^{+\infty} \langle (L - xF)^{n-1} \rangle . L dx \\
&= \frac{1}{\max_\rho a_\rho} (n+1) \int_0^{\infty} \mathrm{Vol}(L - xF) dx. \\
&\geqslant \frac{n+1}{\max_\rho a_\rho} \int_0^{\sqrt[n]{\mathrm{Vol}(L)}} (\mathrm{Vol}(L) - x^n) dx \\
&= \frac{n+1}{\max_\rho a_\rho} \frac{n}{n+1} \mathrm{Vol}(L) \sqrt[n]{\mathrm{Vol}(L)} = \frac{\sqrt[n]{\mathrm{Vol}(L)}}{\max_\rho a_\rho} n \mathrm{Vol}(L), \quad (7.115)
\end{aligned}
$$

where we have used a fact, which states $\langle (L - xF)^{n-1} \rangle$ intersecting with an effective divisor is non-negative since it can be computed by restricted volume (see [71][Theorem B]), for the 4th inequality, (7.99) for the 6th inequality and Lemma 5.2 for the 5th equality.

Thus, we have

$$
\frac{\sqrt[n]{\mathrm{Vol}(L)}}{\max_\rho a_\rho} \leqslant 1 + \frac{\mu}{\mathrm{Vol}(L)} S_L(F) \leqslant 1 + \frac{n}{n+1} \mu \tau_L(F), \quad (7.116)
$$

where we used (5.21). Now, we have shown (7.110). ∎

Remark 7.4. The inequality (7.110) is not invariant under translation. Indeed, if the polytope P is translated to be $P_v := P + v$, then L is invariant as a line bundle. It is easy to that $\mathrm{Vol}(L)$, $\mu(L)$ and $\tau_L(F)$ are invariant. But as a divisor, P_v corresponds to its linear equivalent divisor $L_v := L + \mathrm{div}(\chi^v)$. Then the coefficient of L_v is $a_\rho - \langle u_\rho, v \rangle$. Thus, $\max_\rho a_\rho$ becomes $\max_\rho (a_\rho - \langle u_\rho, v \rangle)$.

Proposition 7.4. As the above Remark, then

$$
\phi(v) := \max_\rho (a_\rho - \langle u_\rho, v \rangle) \quad (7.117)
$$

is bounded from below.

Proof. If $v \in P$, then $\phi(v)$ has a lower bound since P is compact. If $v \notin P$, there exist some ρ such that

$$
\langle u_\rho, v \rangle < -a_\rho, \quad (7.118)
$$

i.e.,

$$
a_\rho - \langle u_\rho, v \rangle > 2a_\rho. \quad (7.119)
$$

We denote $a_{\rho_0} := \min_\rho a_\rho$. One has

$$
\phi(v) \geqslant a_\rho - \langle u_\rho, v \rangle > 2a_\rho \geqslant 2a_{\rho_0}. \quad (7.120)
$$

This finishes the proof. ∎

Thus, we take the infimum in the R.H.S. of (7.110) to get a translation invariant upper bound of the volume.

Example 7.1. We consider

$$P := \{m \in \mathbb{R}^2 | \langle m, u_\rho \rangle \geqslant -a_\rho,\ \rho = 1, 2, 3, 4\}. \tag{7.121}$$

We may assume that $0 < a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Denote $\{v_1, v_2, v_3, v_4\}$ the vertices of P and $\ell_i := \overline{v_i v_{i+1}}$ the edges of P. By Proposition 7.4, we know

$$\phi(v) > 2a_1, \text{ for } v \notin P. \tag{7.122}$$

It is easy to see

$$\phi(v) = 2a_4, \text{ for } v \in \ell_4 \cap P.$$

Remark 7.5. When $L = -K_X$ (toric Fano variety), Theorem 7.6 recovers the volume inequality of the toric case in [50].

Indeed, in this case, we have $\max_\rho a_\rho = 1$ and $\mu = 1$. We take valuation $v = \mathrm{ord}_F$, then we have

$$1 \leqslant \delta(X) \leqslant \frac{n\mathrm{Vol}(L)}{S_L(F)}, \tag{7.123}$$

i.e.,

$$\frac{S_L(F)}{\mathrm{Vol}(L)} \leqslant n. \tag{7.124}$$

By (7.116), we obtain

$$\mathrm{Vol}(L) \leqslant (n+1)^n. \tag{7.125}$$

From this, we also have

$$\frac{\sqrt[n]{\mathrm{Vol}(L)}}{\max_\rho a_\rho} \leqslant 1 + \frac{n\mu(L)}{\delta(L)}. \tag{7.126}$$

Remark 7.6. In the computation (the 5th equality) of the proof of Theorem 7.6, we have

$$n\mathrm{Vol}(L) + n\mu S_L(F) \geqslant \frac{n+1}{\max_\rho a_\rho} S_L(F), \tag{7.127}$$

i.e.,

$$n\mathrm{Vol}(L) \geqslant \left(\frac{n+1}{\max_\rho a_\rho} - n\mu(L)\right) S_L(F). \tag{7.128}$$

When the coefficient of R.H.S. of (7.128) is positive (more general, non-negative), it is nothing but the Zhou-Zhu's bound [102][Theorem 0.1],

$$n\mu(L) \leqslant \frac{n+1}{a_\rho}, \text{ for all } \rho. \tag{7.129}$$

Proposition 7.5. Let (X, L) be a polarized toric variety of dimension n. Assume (X, L) is valuatively seimstable, then we have

$$\delta(L) + n\mu(L) - \frac{n+1}{\max_\rho a_\rho} \geqslant 0. \tag{7.130}$$

Proof. Let F be a toric prime divisor, we have

$$\beta_L(F) \geqslant 0. \tag{7.131}$$

By a similar computation (7.128), one obtains

$$A_X(F)\mathrm{Vol}(L) \geqslant \left(\frac{n+1}{\max_\rho a_\rho} - n\mu\right) S_L(F), \tag{7.132}$$

i.e.,

$$\frac{n+1}{\max_\rho a_\rho} - n\mu \leqslant \frac{A_X(F)\mathrm{Vol}(L)}{S_L(F)}. \tag{7.133}$$

By taking the infimum in the R.H.S., we have

$$\frac{n+1}{\max_\rho a_\rho} - n\mu \leqslant \delta(L). \tag{7.134}$$

∎

We consider a function $f : P_L \to \mathbb{R}$ defined by

$$f(m) := \min_\rho \ell_\rho(m), \tag{7.135}$$

where $\ell_\rho(m) := a_\rho + \langle m, u_\rho \rangle$ is the affine linear function.

For any $m \in M_\mathbb{R}$, then one has $L \sim \sum_{\rho \in \Sigma(1)} \ell_\rho(m) D_\rho$. Furthermore, $L \sim \sum_\rho c_\rho D_\rho$ if and only if there exists a $m \in M_\mathbb{R}$ such that $c_\rho = \ell_\rho(m)$.

Proposition 7.6. Let (X, L) be a polarized toric variety of dimension n. If it satisfies

$$\delta(L) + n\mu(L) \geqslant \frac{n+1}{b_L}, \tag{7.136}$$

where $b_L := \max_{P_L} f$. Then (X, L) is valuatively semistable.

Proof. Let F be any toric prime divisor over X. We may assume that $0 \in P_L$. Then we have

$$\begin{aligned}\beta_L(F) &= A_X(F)\mathrm{Vol}(L) + n\mu(L)S_L(F) - n\int_0^{+\infty} \langle (L - xF)^{n-1} \rangle.(\sum_\rho D_\rho)dx \\ &= A_X(F)\mathrm{Vol}(L) + n\mu(L)S_L(F) - \frac{1}{f(m)}n\int_0^{+\infty} \langle (L - xF)^{n-1} \rangle.(\sum_\rho f(m)D_\rho)dx\end{aligned}$$

$$
\begin{aligned}
&\geqslant A_X(F)\mathrm{Vol}(L)+n\mu(L)S_L(F)-\frac{1}{f(m)}n\int_0^{+\infty}\langle(L-xF)^{n-1}\rangle.(\sum_\rho \ell_\rho(m)D_\rho)dx\\
&= A_X(F)\mathrm{Vol}(L)+n\mu(L)S_L(F)-\frac{n+1}{f(m)}S_L(F)\\
&\geqslant \left(\delta(L)+n\mu(L)-\frac{n+1}{f(m)}\right)S_L(F).
\end{aligned}
\tag{7.137}
$$

Thus, if

$$
\delta(L)+n\mu(L)\geqslant\frac{n+1}{f(m)},
\tag{7.138}
$$

then (X,L) is valuative semistable.

Note that

i) $f(m)=0$ for $m\in\partial P_L$;

ii) $f\geqslant 0$ on P_L;

iii) f is continuous.

Thus, there exists some point $m_0\in\mathring{P}_L$ such that $f(m_0)=\max_{P_L}f=b_L$. By taking $m=m_0$ in (7.138), We finish the proof. ∎

Remark 7.7. (i) b_L is translation invariant.

(ii) Proposition 7.6 gives a sufficient condition of valuative semistability similar with Zhou-Zhu [102]. By Dervan-Legendre [34][Theorem 1.2], we know that the condition (7.136) implies Futaki invariant vanishing.

Lemma 7.3. Let (X,L) be a polarized toric variety of dimension n. It satisfies

$$
\delta(L)\geqslant\frac{n+1}{\min_\rho a_\rho}-\frac{n+1}{\max_\rho a_\rho},
\tag{7.139}
$$

then (X,L) is valuatively semistable.

Proof. We divide into two cases.

(i) If

$$
0<n\mu(L)<\frac{n+1}{\max_\rho a_\rho}.
\tag{7.140}
$$

By Zhou-Zhu's result, (X,L) is K-stable, which implies that (X,L) is valuative semistable.

(ii) If

$$
n\mu(L)\geqslant\frac{n+1}{\max_\rho a_\rho}.
\tag{7.141}
$$

By assumption, it follows that

$$
n\mu(L)\geqslant\frac{n+1}{\min_\rho a_\rho}-\delta(L).
\tag{7.142}
$$

By Proposition 7.6, then (X, L) is valuative semistable.

∎

A natural question is what is happen when

$$0 < \delta < \frac{n+1}{\min_\rho a_\rho} - \frac{n+1}{\max_\rho a_\rho} \tag{7.143}$$

and

$$\frac{n+1}{\max_\rho a_\rho} < n\mu < \frac{n+1}{\min_\rho a_\rho} - \delta(L). \tag{7.144}$$

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Acknowledgements

This work would not have been possible without all the help and support provided by others. During my time here in Beijing and in particular at Tsinghua University and the Institute of Mathematics, Chinese Academy of Sciences, I have had the honor of meeting a lot of interesting people, many of whom became friends. To all of you, I hereby say: *Thank You for everything! This would not have been possible without you!* However, there are a couple of individuals who deserve special mention.

First of all, I would like to thank my supervisor **Prof. Akito Futaki** for his constant help and guidance in last four years. Prof. Futaki taught me not only mathematics, but also how to become an independent mathematician. When I was a master in Chinese Academy of Science, Prof. Futaki had already started to guide me on Kähler Geometry. He brought me into the fascinating field of K-stability and guided me to finish my first paper. During my PhD, Prof. Futaki always supported and encouraged me to do the independent research and consideration. He always explained many interesting ideas to me and discussed patiently about my immature ideas.

Next, I would like to thank **Prof. Xiaohua Zhu**. I always attended the seminar of Prof. Zhu and learned everything to do an interesting research, including finding the significant and absorbing questions, the rigorous approach, and many interesting ideas, etc. Discussion with him broadens my knowledge. Prof. Zhu has given me extraordinary help, even as a co-supervisor.

Next, I hope to give my thanks to **Prof. Xiangyu Zhou**. I learned pluripotential theory and analytic geometry from the seminar of Prof. Zhou. Prof. Zhou guided and collaborated my third paper with his PhD students Zhuo Liu and Hui Yang. Prof. Zhou always tolerated our some naive errors and shared many amusing ideas. Prof. Zhou did not mind my weak foundation about several complex variables, and patiently explained the basic concepts.

Then I hope to accord my thanks to **Sébastien Boucksom, Mattias Jonsson, Tamás Darvas, Ruadhaí Dervan, Eveline Legendre, Vestislav Apostolov**. When I entered the field of K-stability, I sent many emails to ask them some basic questions about their papers. They always answered my questions patiently and friendly.

Furthermore, I want to thank the other mathematicians that I have worked with or discussed with: **Jian Xiao, Chi Li, Yanir Rubinstein, Mingchen Xia, Kewei Zhang, Chang Li, Yan Li, Bin Zhou, Jianchun Chu, Wangjian Jian, Abdellah Lahdili, Zhuo Liu, Hui Yang, Xiaowei Jiang, Eiji Inoue, Binggang Qu, Weijun Zhang, Zheng Xu, Xiangfei Li**. Also my friend: **Ganghui Zhang, Yuting Liu, Liang Zhang, Pengcheng Li, Xinyu Zhu, Ziyi Zhao**.

A special thanks goes to my friend **Xiangfei Li** and **Zixin Ye** for helping me improve my English grammar.

Last but not least, I would like to thank **my families** for their selfless support and environment they created, in which I was able to focus on mathematics. Without their nurture and upbringing, I would not have been able to do my PhD.

声　明

本人郑重声明：所呈交的学位论文，是本人在导师指导下，独立进行研究工作所取得的成果。尽我所知，除文中已经注明引用的内容外，本学位论文的研究成果不包含任何他人享有著作权的内容。对本论文所涉及的研究工作做出贡献的其他个人和集体，均已在文中以明确方式标明。

签　名：＿＿＿＿＿＿ 日　期：＿＿＿＿＿＿

Resume, Achievements during the PhD

Resume

1995.02.28 Born in Gangu County, Gansu Province.

2013.09–2017.06, B.S. in Mathematics and Applied Mathematics, Lanzhou University.

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Achievements during the PhD

This thesis is based on the following paper:

- Liu Y. Openness of uniformly valuative stability on the Kähler cone of projective manifolds. Math. Z., 2023, 303 (2): 52.

Other publication:

- Liu Y. Uniform K-stability and conformally Kähler, Einstein-Maxwell geometry on toric manifolds. Tohoku Math. J. (2), 2022, 74 (1): 1–21.

Preprint:

- Liu Y, Liu Z, Yang H, Zhou X. A Le Potier-type isomorphism theorem for holomorphic vector bundles with strongly Nakano positive singular hermitian metrics. preprint.

Comments from Thesis Supervisor

After the celebrated work of the existence of zero Ricci curvature Kähler metric (Calabi-Yau metric), and negative Kähler-Einstein metrics by Shing-Tung Yau (1976), the existence problems of positive Kähler-Einstein metrics and constant scalar curvature Kähler (cscK for short) metrics remained to be studied. In both cases there have been known obstructions, and Yau conjectured that the existence should be equivalent to certain stability condition in algebraic geometry. This conjecture is called the Yau conjecture.

This conjecture was confirmed for Fano manifolds, stating that the existence of Kähler-Einstein metrics is equivalent to K-polystability, in the papers published in 2015 by Chen-Donaldson-Sun (J. AMS) and Tian (Comm. Pure Appl. Math.). They used the method of Kähler-Einstein metrics with cone angle along divisors and Cheeger-Colding theory on Gromov-Hausdorff convergence.

Later alternate proofs were given by Datar-Székelyhidi (2016 GAFA) using the continuity method of Yau, and by Chen-Sun-Wang (2018 Geom. Top.) using the Kähler-Ricci flow. But the both works use Gromov-Hausdorff convegence, which can not be used for the cscK case.

An alternate proof when the automorphism group is discrete was given by Berman-Boucksom-Jonsson (2021 J.AMS) without using Gromov-Hausdorff convergence. In this work, the stability condition is described in terms of non-Archimedean functionals, which mean the invariants which describes the infinitesimal behavior at ∞ (i.e. slope) of the relevant functionals (due to Mabuchi and Ding) along geodesics. Their result is referred to as uniform K-stability. More recently, Chi Li modified the proof of Berman-Boucksom-Jonsson to apply to the case when the automorphism group is not discrete (2022 Invent. Math.). The result in this case is referred to as G-uniform stability. When G contains the maximal torus of the automorphism group, G-uniform stability is equivalent to K-polystability. This equivalence is non-trivial and follows from the work of Liu-Xu-Zhuang (Preprint 2021 to appear in Ann. of Math.), which is described next.

So far the existence results were treated, but there were developments from other direction, namely the problem of how to check the K-polystability and to apply to the moduli problem of K-polystable Fano varieties, called the K-moduli. This development uses the recent development of the Minimal Model Program. This connection was first pointed out by Yuji Odaka, and further elaborated by Chenyang Xu, Chi Li, Xiaowei Wang, Yuchen Liu, Ziquan Zhuang, Harold Blum, Kento Fujita and others. The above mentioned work of Liu-Xu-Zhuang gives a final result to assure that the K-moduli of K-polystable Fano varieties are algebraic. It should also be mentioned that K. Fujita (2019 J. Reine Angew. Math.) and C. Li (2017 Duke Math.) gave a criterion called the valuative criterion for K-stability and Fujita-Odaka formulated δ-invariant to check K-stability. These works played the key role to apply birational geometry. More recently, Kewei Zhang defined *analytic δ-invariant* in differential geometric way, showed that δ-invariant is equal to the analytic δ-invariant, and revealed the relationship between the δ-invariant and the Moser-Trudinger inequality and α-invariant (Preprint arXiv:2102.02438, to appear in J. European Mathematical Society).

As for the cscK case, the so-called easy direction “existence implies K-polystability” was proved by Berman-Darvas-Lu (2020 Ann. Sci. Éc. Norm. Supér.). The result is not easy at all, and the paper is well written. Fujita-Odaka’s delta invariant was extended to the cscK case by Dervan-Legendre and used to define a notion called the valuative stability (published on line 2020, Math. Ann.). Because of the lack of results from the minimal model program this valuative stability for cscK metrics only gives a necessary condition for the existence. The present thesis by Yaxiong Liu proves the openness of the uniform valuative stability in the ample cone. Further he found two applications of this result. One is the valuative criterion for the existence of J-equation where the J-equation has been considered by Donaldson and studied extensively by many mathematicians. Another is the upper bound of polarized toric varieties. These results of Yaxiong Liu go along the main stream of the field , and are considered as an important contribution for further future research.

Written by Akito Futaki

March 12, 2023

Resolution of Thesis Defense Committee

流形上的典则度量和各种稳定性之间的关系是复几何研究的核心问题之一。近十几年来，这一问题的研究取得了重大进展。这些稳定性与一类几何不变量的正性紧密相关，如 Li Chi, Fujita 等人提出了用代数簇的除子来计算 δ, β 等几何不变量。本博士论文围绕着一般极化流形上的常数量曲率度量和赋值稳定性，对包括扰动下一致赋值稳定性是否保持、极化簇 K-稳定性的赋值稳定性判别法、赋值稳定性在 J 方程中的应用等几个问题展开研究。选题具有重要的理论意义并取得了创新性的研究成果，主要包含以下几个方面：

(1) 证明了正规代数簇的一致赋值稳定性轨迹是丰沛锥的一个开子锥；

(2) 得到了一致赋值稳定性阈值函数的连续性；

(3) 建立了 J-方程解存在性的赋值判别法，并给出了半稳定极化环簇体积的上界估计。

本论文写作规范，作者对相关文献的掌握比较全面，体现了其扎实的几何分析功底和具备独立科学研究的能力。定理的证明推理严谨，富有创新性。答辩过程中叙述清楚，回答问题准确。经过答辩委员会认真讨论，无记名投票表决一致通过刘亚雄的论文答辩，认为这是一篇优秀的博士论文，建议授予刘亚雄理学博士学位。