# A LE POTIER-TYPE ISOMORPHISM TWISTED WITH MULTIPLIER SUBMODULE SHEAVES

YAXIONG LIU, ZHUO LIU, HUI YANG AND XIANGYU ZHOU

Abstract. In this paper, we obtain a Le Potier-type isomorphism theorem twisted with multiplier submodule sheaves, which relates a holomorphic vector bundle endowed with a strongly Nakano semipositive singular Hermitian metric to the tautological line bundle with the induced metric. As applications, we obtain a Kollár-type injectivity theorem, a Nadel-type vanishing theorem, and a singular holomorphic Morse inequality for holomorphic vector bundles and so on.

## 1. Introduction

Let $E$ be a holomorphic vector bundle of rank $r$ over a complex manifold $X$. Consider the projectivized bundle $\mathbb{P}(E^*)$ of the dual bundle $E^*$ and its tautological line bundle $L_E := \mathcal{O}_{\mathbb{P}(E^*)}(1)$. Let $\pi$ be the induced projection from $\mathbb{P}(E^*)$ to $X$. It is well-known that $\pi_* L_E^m = S^m E$ for $m \geq 1$ where $S^m E$ is the $m$-th symmetric tensor power of $E$. On the one hand, Kobayashi–Ochiai [29] obtained isomorphisms

$$K_{\mathbb{P}(E^*)/X} = (L_E)^{-r} \otimes \pi^* \det E \tag{1}$$

and

$$H^q(X, W \otimes S^m E) = H^q(\mathbb{P}(E^*), \pi^* W \otimes L_E^m),$$

where $W$ is a holomorphic vector bundle over $X$. On the other hand, via using the spectral sequence, Le Potier [30] gave the following isomorphism theorem which establishes a connection between the cohomology of vector bundles and that of line bundles:

$$H^q(X, \Omega_X^p \otimes E) = H^q(\mathbb{P}(E^*), \Omega^p_{\mathbb{P}(E^*)} \otimes L_E).$$

By the way, Schneider [41] gave a simpler proof of Le Potier's isomorphism by Künneth's formula and Bott's vanishing.

In this paper, we discuss the above theorems in the setting of *singular hermitian metrics* and establish a general Le Potier-type isomorphism theorem twisted with *multiplier ideal* (*submodule*) *sheaves* associated to *singular Hermitian metrics* which is a generalization of both of the above isomorphism theorems.

Recall that a singular Hermitian metric $h$ on $E$ is a measurable map from $X$ to the space of non-negative Hermitian forms on the fibers satisfying $0 < \det h < +\infty$ almost everywhere. The metric $h$ is called *Griffiths semi-positive*, denoted by $(E, h) \geq_{\mathrm{Grif}} 0$, if $|u|^2_{h^*}$ is plurisubharmonic for any local holomorphic section $u$ of $E^*$. And $h$ is called *Griffiths positive*, denoted by $(E, h) >_{\mathrm{Grif}} 0$, if locally there exists a smooth strictly plurisubharmonic function $\psi$ such that $he^{\psi}$ is Griffiths semi-positive. For any singular Hermitian metric $h$ on

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$E$, it induces a singular Hermitian metric $h_L$ on $L_E$ by the quotient morphism $\pi^*E \to L_E$ . Then it follows from Proposition 2.4 that the $(L_E, h_L) \geq_{\text{Grif}} 0$ as soon as $(E, h) \geq_{\text{Grif}} 0$.

As a tool for characterizing the singularities of Griffiths semi-positive singular Hermitian metrics on holomorphic line bundles, the multiplier ideal sheaf has played an important role in several complex variables and complex algebraic geometry. In [8], M. A. de Cataldo defined the multiplier submodule sheaf $\mathcal{E}(h)$ of $\mathcal{O}(E)$ associated to a singular Hermitian metric $h$ on a holomorphic vector bundle $E$ as follows:

$$\mathcal{E}(h)_x := \{u \in E_x : |u|_h^2 \text{ is integrable in some neighborhood of } x\}.$$

When $E$ is a holomorphic line bundle, $\mathcal{E}(h)$ is simply $\mathcal{O}(E)\otimes\mathcal{I}(h)$, where $\mathcal{I}(h)$ is the multiplier ideal sheaf associated to $h$.

It's well-known that the multiplier ideal sheaf associated to a pseudo-effective line bundle satisfies some basic properties: coherence, torsion-freeness, Nadel vanishing theorem and so on. Nevertheless, it remains uncertain whether Griffiths semi-positivity implies the coherence of multiplier submodule sheaves. In more recent developments, Inayama [25] has showed that in cases where the unbounded locus of $\det h$ is isolated, the Griffiths semi-positivity of $(E, h)$ indeed leads to the coherence of $\mathcal{E}(h)$. Similarly, Zou [54] has demonstrated that the coherence of $\mathcal{E}(h)$ is a consequence of the Griffiths semi-positivity of $(E, h)$ when $\det h$ have analytic singularities.

To derive further insights into multiplier submodule sheaves, we must delve into notions of positivity that surpass Griffiths positivity. In [47], Wu introduced a new positivity for singular Hermitian metrics on holomorphic vector bundles and showed that this positivity is stronger than Nakano positivity for smooth Hermitian metrics.

**Definition 1.1.** We say that a singular Hermitian metric $h$ is *strongly Nakano semi-positive*, denoted by $(E, h) \geq_{\text{SNak}} 0$, if $(E, h) \geq_{\text{Grif}} 0$ and

$$(L_E^{r+1} \otimes \pi^* \det E^*, h_L^{r+1} \otimes \pi^* \det h^*) \geq_{\text{Grif}} 0.$$

Moreover, we say that a singular Hermitian metric $h$ is *strongly Nakano positive*, denoted by $(E, h) >_{\text{SNak}} 0$, if locally there exists a smooth strictly plurisubharmonic function $\psi$ such that $(E, he^{\psi}) \geq_{\text{SNak}} 0$.

**Remark 1.2.** (i) Our definition may appear slightly different from Wu's article, yet it coincides as long as we presume prior Griffiths semi-positivity of $h$.

(ii) Let $M$ be a holomorphic line bundle over $X$ endowed with a singular Hermitian metric $h_M$. By abuse of notation, we always denote by

$$(E, h) \geq_{\text{SNak}} \sqrt{-1}\Theta_{(M,h_M)}$$

if $(E \otimes M^*, h \otimes h_M^*) \geq_{\text{Grif}} 0$ and

$$(L_E^{r+1} \otimes \pi^*(\det E^* \otimes M^*), h_L^{r+1} \otimes \pi^*(\det h^* \otimes h_M^*)) \geq_{\text{Grif}} 0.$$

(iii) It is easy to check that $(E, h) \geq_{\text{Grif}} 0$ (resp. $>_{\text{Grif}} 0$) implies $(E\otimes\det E, h\otimes\det h) \geq_{\text{SNak}} 0$ (resp. $>_{\text{SNak}} 0$) and strongly Nakano positivity is corresponding to Griffiths positivity when $E$ is of rank one. It is well-known that an ample vector bundle over Riemann surface must admit a Griffiths positive smooth Hermitian metric ([45, 6]). However, it does not admit necessarily a strongly Nakano positive smooth Hermitian metric (see Example 4.5).

(iv) Notice that when a smooth Hermitian metric $h$ is strongly Nakano (semi-)positive, it gives rise to a smooth and (semi-)positive metric $h_F := h_L^{m+r} \otimes \pi^*(\det h^*)$ on $\widetilde{F} := K^*_{\mathbb{P}(E^)/X} \otimes L_E^m$, where $h_L$ is the metric induced by $h$ on $L_E$. Remarkably, Berndtsson [1] showed that the $L^2$-metric induced by $h_F$ (initially introduced by Narasimhan–Simha [38]) on $\pi_*(K_{\mathbb{P}(E^*)/X} \otimes F) = S^m E$ is Nakano (semi-)positive. Furthermore, Liu–Sun–Yang [31, Theorem 7.1] showed that the $L^2$-metric is just a constant multiple of $S^m h$. Thus strong Nakano positivity implies Nakano positivity for smooth Hermitian metrics.

(v) We will show that strong Nakano positivity implies Nakano positivity too for singular Hermitian metrics (See Proposition 3.4).

Wu proposed [47, Problem 5.1] on the potential equivalence between Nakano positivity and strongly Nakano positivity. We consider the decomposable vector bundle of rank two over Riemann surface, whose dual projectivized bundle is the so-called *Hirzebruch-like ruled surfaces* and prove that it is Nakano (semi-)positive, but not *strongly* Nakano (semi-)positive (see Example 4.5). This gives a negative answer to Wu's problem.

With these notions, we obtain a Le Potier-type isomorphism theorem for holomorphic vector bundles with strongly Nakano semi-positive singular Hermitian metrics.

**Theorem 1.3** (Main Theorem)**.** *Let $E$ be a holomorphic vector bundle of rank $r$ over a complex manifold $X$. Assume that $(E, h) \geq_{\mathrm{SNak}} 0$.*

(i) *If $W$ is a holomorphic vector bundle over $X$, then for any $q \geq 0$ and $m \geq 1$, we have*
$$H^q(X, W \otimes S^m\mathcal{E}(S^m h)) \simeq H^q(\mathbb{P}(E^*), \pi^* W \otimes L_E^m \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)),$$

(ii) *and for any $p, q \geq 0$, we have*
$$H^q(X, \Omega_X^p \otimes \mathcal{E}(h)) \simeq H^q(\mathbb{P}(E^*), \Omega^p_{\mathbb{P}(E^*)} \otimes L_E \otimes \mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)).$$

**Remark 1.4.** (i) By the formula (1), $L_E^m \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)$ is actually
$$K_{\mathbb{P}(E^*)/X} \otimes L_E^{m+r} \otimes \pi^* \det E^* \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*).$$

(ii) It is well-known that the multiplier ideal sheaf itself does not have the factorial property under proper modifications but obtains it when twisted with the canonical line bundle. This explains our focus on the isomorphism
$$\pi_*(K_{\mathbb{P}(E^*)} \otimes (L_E)^{m+r} \otimes \pi^* \det E^*) = K_X \otimes S^m E$$
rather than $\pi_*(L_E)^m = S^m E$.

(iii) We actually prove the main theorem when there exists locally a smooth function $\psi$ such that $he^{-\psi}$ is strongly Nakano semi-positive. Especially, if $h$ is smooth, then muiltiplier submodule sheaves become trivial and the above theorem becomes Le potier's isomorphism and Kobayashi-Ochiai's isomorphism.

Our proof is different from Le potier's and Schneider's. By using Liu–Sun–Yang's formula [31], we can show that

$$\pi_*(K_{\mathbb{P}(E^*)/X} \otimes L_E^{m+r} \otimes \pi^* \det E^* \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)) \simeq \mathcal{S}^m\mathcal{E}(S^m h), \tag{2}$$

for Griffiths semi-positive singular Hermitian metrics $h$. Therefore, Theorem 1.3-(i) can be deduced from Leray's isomorphism theorem if one can show the vanishing of higher direct image sheaves

$$R^k\pi_*(K_{\mathbb{P}(E^*)/X} \otimes L_E^{m+r} \otimes \pi^* \det E^* \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)). \tag{3}$$

When $h$ is smooth, (3) can be obtained by Künneth's formula and Bott's vanishing. However, the approach fails when twisted with mutiplier submodule sheaves. Noticing that $h_L^{m+r} \otimes \pi^* \det h^*$ is positive along the fiber $\pi^{-1}(x)$ such that $\det h(x)$ is finite, the strongly Nakano semi-positivity of $h$ assures that the higher direct images vanishes (see Lemma 3.6). In fact, we only need to assume that there is locally a smooth function $\psi$ such that $he^{-\psi}$ is Griffiths semi-positive and strongly Nakano semi-positive respectively. In particular, we give a new proof of Le potier's isomorphism.

It is worth mentioning that (2) offers insight into exploring the coherence of $\mathcal{E}(h)$ associated to a Griffiths semi-positive vector bundle $(E,h)$ through an investigation of the coherence of $\mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)$, owing to Grauert's direct image theorem.

**Proposition 1.5** (Proposition 4.1)**.** *Let $(E,h) \geq_{\mathrm{Grif}} 0$.*

(i) *If $\mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)$ is coherent, then $\mathcal{E}(h)$ is coherent.*
(ii) *If $\det h$ has analytic singularities, then $\mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)$ is coherent.*

Similarly, we can also obtain the following isomorphism theorem.

**Theorem 1.6.** *Let $E$ be a holomorphic vector bundle of rank $r$ over a complex manifold $X$. Assume that $(E,h) \geq_{\mathrm{Grif}} 0$,*

(i) *then for $q \geq 0, m \geq 1$, we have*

$$H^q(X, K_X \otimes S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h)) \simeq H^q(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)} \otimes L_E^{m+r} \otimes \mathcal{I}(h_L^{m+r})),$$

(ii) *and for $p, q \geq 0$, we have*

$$H^q(X, \Omega_X^p \otimes \mathcal{E} \otimes \det \mathcal{E}(h \otimes \det h)) \simeq H^q(\mathbb{P}(E^*), \Omega^p_{\mathbb{P}(E^*)} \otimes L_E \otimes \pi^* \det E \otimes \mathcal{I}(h_L^{1+r})).$$

Theorem 1.3 and Theorem 1.6 builds a bridge between the multiplier submodule sheaves and the multiplier ideal sheaves on its dual projectivized bundles. This connection offers a pathway to explore the properties of multiplier submodule sheaves by the well-developed theory of multiplier ideal sheaves, which includes Kollár-type injective theorems ([27, 14, 15, 51]), Nadel-type vanishing theorems ([37, 7, 20, 36]), and holomorphic Morse inequalities ([10, 35, 12]).

Firstly, Zhou–Zhu [51] derived an injective theorem for pseudo-effective line bundles on holomorphically convex manifolds (see Theorem 2.16). Consequently, using Theorem 1.3, we can establish a Kollár-type injective theorem for strongly Nakano positive vector bundles.

**Corollary 1.7** (Theorem 4.6)**.** *Let $(X,\omega)$ be a holomorphically convex Kähler manifold and $(E,h) \geq_{\mathrm{SNak}} 0$. Assume that*

$$(E,h) \geq_{\mathrm{SNak}} b\sqrt{-1}\Theta_{(M,h_M)}$$

*for some $0 < b < +\infty$, and*

$$\sqrt{-1}\Theta_{(M,h_M)} \geq -C\omega$$

*for some constant $C$. Then for any non-zero $s \in H^0(X,M)$ satisfying*

$$\sup_{\Omega} |s|_{h_M} < +\infty$$

*for every $\Omega \Subset X$, we obtain that the map*

$$H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes S^m\mathcal{E} \otimes \mathcal{M}(S^m h \otimes h_M))$$

*is injective for any $q \geq 0$.*

Secondly, a Kawamata–Viehweg–Nadel-type vanishing theorem for a pseudo-effective line bundle $(M,h_M)$ twisted with the multiplier ideal sheaf $\mathcal{I}(h_M)$ was asked as a question by Cao [7], and solved by Guan–Zhou [20] as a consequence of their proof of Demailly's strong openness conjecture. Therefore, by applying Theorem 1.3, we can deduce a Nadel-type vanishing theorem for strongly Nakano semi-positive vector bundles on compact Kähler manifolds.

**Corollary 1.8** (Corollary 4.10). *Let $(E,h) \geq_{\mathrm{SNak}} 0$ be a holomorphic vector bundle of rank $r$ over a compact Kähler manifold $X$ of dimension $n$. then for any $m \geq 1$, we have*

$$H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) = 0$$

*for any* $q \geq n + r - \mathrm{nd}(L_E^{m+r} \otimes \pi^* \det E^*, h_L^{m+r} \otimes \pi^* \det h^*)$.

Notice that if $(E,h) >_{\mathrm{SNak}} 0$ over a compact (possibly non-Kähler) manifold $X$, then $X$ admits a Kähler modification and $\mathrm{nd}(L_E^{m+r} \otimes \pi^* \det E^*, h_L^{m+r} \otimes \pi^* \det h^*) = n + r - 1$. In addition, Corollary 1.7 conforms that cohomology groups remain invariant under Kähler modifications. Hence, we can conclude that

**Corollary 1.9** (Corollary 4.11). *Let $(E,h) >_{\mathrm{SNak}} 0$ be a holomorphic vector bundle over a compact (possibly non-Kähler) manifold $X$, then for any $m \geq 1$, we have*

$$H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) = 0$$

*for $q \geq 1$.*

If $(E,h)$ is merely Griffiths (semi-)positive, applying Theorem 1.6, we can also derive the corresponding injectivity and vanishing theorems for $\mathcal{S}^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h)$ by a similar argument.

Thirdly, Bonavero [5] obtained Demailly-type holomorphic Morse inequalities twisted with multiplier ideal sheaves associated to singular metrics with analytic singularities. Consequently, employing Theorem 1.6, we can also conclude singular holomorphic Morse inequalities for holomorphic vector bundles.

**Corollary 1.10** (Theorem 4.15). *Let $(X,\omega)$ be a compact complex manifold of dimension $n$ and $(E,h)$ a holomorphic vector bundle endowed with a singular Hermitian metric. Assume that there is a smooth function $\phi$ such that $(E, he^{-\phi}) \geq_{\mathrm{Grif}} 0$ and the induced metric $h_L$ on $L_E$ over $\mathbb{P}(E^*)$ has analytic singularities. Let $V$ be a holomorphic vector bundle on $X$. Then for $0 \leq q \leq n$, we have*

$$\begin{aligned}&h^q(X, V \otimes S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h))\\ \leq&\mathrm{rank}(V)\frac{(m+r)^{n+r-1}}{(n+r-1)!}\int_{\mathbb{P}(E^*)(q)}(-1)^q(\sqrt{-1}\Theta_{(L_E,h_L)})^{n+r-1} + o(m^{n+r-1}),\end{aligned}$$

*and*

$$\begin{aligned}&\sum_{j=0}^{q}(-1)^{q-j}h^j(X, V \otimes S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h))\\ \leq&\mathrm{rank}(V)\frac{(m+r)^{n+r-1}}{(n+r-1)!}\int_{\mathbb{P}(E^*)(\leq q)}(-1)^q(\sqrt{-1}\Theta_{(L_E,h_L)})^{n+r-1} + o(m^{n+r-1})\end{aligned}$$

*as $m \to +\infty$ (see* Section 4.5 *for the definitions of $\mathbb{P}(E^*)(q)$ and $\mathbb{P}(E^*)(\leq q)$).*

Finally, based on Theorem 1.6 and Darvas–Xia's [12] asymptotic formula (Theorem 4.18) for the volumes of pseudo-effective line bundles, we obtain the following asymptotic formula for Griffiths positive vector bundles.

**Corollary 1.11** (Theorem 4.20). *Let $X$ be a compact Kähler manifold and $(E,h) \geq_{\mathrm{Grif}} 0$. Then we have*

$$\lim_{m\to\infty}\frac{1}{m^{n+r-1}}h^0(X, S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h))$$

$$= \frac{1}{(n+r-1)!}\int_{\mathbb{P}(E^*)}(\sqrt{-1}\Theta_{(L,P[h_L]_{\mathcal{I}})})^{n+r-1}.$$

*where* $P[h_L]_{\mathcal{I}} \in \mathrm{Psh}(X, L)$ *is the so-called* $\mathcal{I}$*-model envelope of positive metric* $h_L$ *(see* Section 4.5*) and* $(\sqrt{-1}\Theta_{(L,P[h_L]_{\mathcal{I}})})^{n+r-1}$ *is the Monge-Ampère measure of* $P[h_L]_{\mathcal{I}}$ *(see* Section 4.4*).*

In particular, we obtain an asymptotic inequality for vector bundles with Griffiths positive singular Hermitian metrics on compact manifolds, which can be viewed as a generalization of the classical result of big line bundles.

**Theorem 1.12** (Theorem 4.21)**.** *Let* $(E, h) \geq_{\mathrm{Grif}} 0$ *be a holomorphic vector bundle of rank* $r$ *over a compact (possibly non-Kähler) manifold* $X$ *of dimension* $n$*. Then we have*

$$\lim_{m\to\infty}\frac{1}{m^{n+r-1}}h^0(X, S^m\mathcal{E}(S^m h)) \geq \frac{1}{(n+r-1)!}\int_{\mathbb{P}(E^*)}\left(\sqrt{-1}\Theta_{(L_E,h_L)}\right)^{n+r-1}.$$

This article is organized as follows.

- In Section 2,we revisit fundamental concepts regarding the positivity of singular Hermitian metrics and $L^2$-metrics on the direct image sheaves.
- In Section 3, we present a proof of our main Theorem 1.3.
- In Section 4, we explore various applications from Theorem 1.3 via connecting with results of pseudo-effective line bundles.

## 2. Preliminaries

The primary aim of this preliminary section is to provide a comprehensive overview of the fundamental theory concerning the positivity of metrics on holomorphic vector bundles and Berndtsson's $L^2$-metrics. Additionally, we revisit classical results in complex analytic geometry which are crucial for our proof.

### 2.1. **Notation and convention.**

- Throughout the paper, unless explicitly specified otherwise, all metrics are considered singular Hermitian metrics.
- We fix $X$ as a complex manifold of dimension $n$ and $E$ as a holomorphic vector bundle of rank $r$ on $X$. When we say $(X, \omega)$ is a complex (resp. Hermitian, Kähler) manifold, it means that $\omega$ is a smooth (resp. Hermitian, Kähler) metric on $X$.
- For the convenience of writing, we do not differentiate between the Hermitian metrics of line bundles and its (global) metric weights.
- To maintain consistency in notation, we denote metrics on holomorphic vector bundles as $h$, metrics of holomorphic line bundles as $\varphi$, and the $L^2$-metric of $E$ induced by the metric $\varphi$ of $L_E$ as $h_\varphi$.
- We consistently use $L_E$ to represent the tautological line bundle $\mathcal{O}_E(1)$ of $E$ over $\mathbb{P}(E^*)$.
- The metric of $L_E$ induced by the metric $h$ is denoted as $h_L$. Additionally, in cases where no confusion arises, we use $\varphi_h$ for the metric weight of $h_L$.

### 2.2. $L^2$**-metrics on the direct image sheaves.** 

Recall some notations from [2, §2.1].

Let $F$ be a holomorphic line bundle over a complex manifold $Y$ of dimension $\ell$ and $\{U_j\}$ be an open covering of the manifold such that $F$ is trivial on each $U_j$. A section $s$ of $F$ consists of complex-valued functions $s_j$ on $U_j$ satisfying $s_j = g_{jk}s_k$, where $g_{jk}$ represents the transition functions of the bundle. A metric (weight) is a set of real-valued functions $\psi^j$ on $U_j$ such that the globally well-defined expression

$$|s|^2_\psi := |s|^2 e^{-\psi} := |s_j|^2 e^{-\psi^j}$$

holds. We denote $\psi$ to represent the collection $\psi^j$.

A metric $\psi$ on $F$ induces an $L^2$-metric on the adjoint bundle $K_Y \otimes F$. A section $\xi$ of $K_Y \otimes F$ can be locally expressed as

$$\xi = dz \otimes s,$$

where $dz = dz^1 \wedge \cdots \wedge dz^\ell$ for some local coordinate of $Y$ and $s$ is a local section of $F$. We define

$$|\xi|^2 e^{-\psi} := c_n dz \wedge d\overline{z} |s|^2_\psi,$$

which represents a (global) volume form on $Y$. The $L^2$-norm of $\xi$ is given by

$$\|\xi\|^2 := \int_Y |\xi|^2 e^{-\psi}.$$

It is important to note that the $L^2$-norm only depends on the metric $\psi$ on $F$, but does not involve any choice of metrics on the manifold $Y$.

Consider a holomorphic proper fibration $\pi : Y \to X$ between complex manifolds $X$ and $Y$, and a holomorphic line bundle $F$ on $Y$ with a Griffiths semi-positive smooth metric $\varphi$. Assume that $\pi_*(K_{Y/X} \otimes F)$ is locally free and thus $(\pi_*(K_{Y/X} \otimes F))_x = H^0(Y_x, K_{Y_x} \otimes F|_{Y_x})$. Berndtsson introduced an $L^2$-metric $h_\varphi$ on $\pi_*(K_{Y/X} \otimes F)$ by defining

$$|u|^2_{h_\varphi} := \int_{Y_x} |u|^2 e^{-\varphi},$$

where $u \in (\pi_*(K_{Y/X} \otimes F))_x$ and $Y_x = \pi^{-1}(x)$.

**Theorem 2.1** ([1]). *Assume that $h_F$ is smooth positive, then the $L^2$-metric on $\pi_*(K_{Y/X} \otimes F)$ is Nakano positive.*

Now, let us assume $h$ to be a Griffiths positive singular Hermitian metric on $E$. Then $h$ induces a singular metric $h_L$ on $L_E$. Consider a coordinate $(z_1, \ldots, z_n)$ around $x \in X$ and any holomorphic frame $e = \{e_1, \ldots, e_r\}$ of $E$. The dual frame is $e^* = \{e_1^*, \ldots, e_r^*\}$ of $E^*$. Let $W^1, \ldots, W^r$ be the coordinate of $E^*$ with respect to $e^*$. The homogeneous coordinate is represented as $[W^1 : \cdots : W^r]$. On $\{W^1 \neq 0\}$, $w^j := W^j/W^1, j = 2, \ldots, n$ is a local coordinate along fibers.

For any $a \in \mathbb{P}(E_x^*) \backslash \{W_1 = 0\}$, $(z_1, \ldots, z_n, w^2, \ldots, w^r)$ defines a local coordinate on $\mathbb{P}(E^*)$ near $a$. We can define a local non-vanishing holomorphic section of $\mathcal{O}_E(-1) = \mathcal{O}_E(1)^* \subset \pi^* E^*$ over $\mathbb{P}(E^*)$ as

$$s : (z, w) \mapsto \sum_{1 \le \alpha \le r} w^\alpha e_\alpha^* = e_1^* + \sum_{2 \le \alpha \le r} w^\alpha e_\alpha^*.$$

The metric $h_L^*$ on $\mathcal{O}_E(-1)$ induced by $h$ is given by

$$|s|^2_{h_L^*} = \sum_{1 \le \alpha, \beta \le r} w^\alpha \overline{w^\beta} \left\langle e_\alpha^*, e_\beta^* \right\rangle_{h^*} = \sum_{1 \le \alpha, \beta \le r} w^\alpha \overline{w^\beta} h^*_{\alpha\beta}.$$

Hence the metric $h_L$ on $L_E$ is obtained as

$$h_L = \frac{1}{\sum_{1 \le \alpha, \beta \le r} w^\alpha \overline{w^\beta} h^*_{\alpha\beta}}. \tag{4}$$

Additionally, let $h$ be smooth and $\{e_\alpha\}$ be a normal frame of $E$ at $x \in X$. Then the curvature of $h$ at $x$ is given by

$$\Theta_{(E,h)} = c_{ij\alpha\beta} dz^i \wedge d\overline{z}^j \otimes e_\alpha^* \otimes e_\beta,$$

satisfying $\overline{c}_{ij\alpha\beta} = c_{ji\beta\alpha}$.

At any point $b \in \mathbb{P}(E_x^*)$ represented by a vector $\sum b_\alpha e_\alpha^*$ of norm 1, then the curvature of $h_L$ is derived as

$$\begin{aligned}\Theta_{(L_E,h_L)}(b) =& \sum c_{ij\beta\alpha} b_\alpha \overline{b_\beta} dz^i \wedge d\overline{z}^j + \sum dw^\lambda \wedge d\overline{w}^\lambda \\ =& \langle -\Theta(h^*)b, b\rangle + \sum dw^\lambda \wedge d\overline{w}^\lambda.\end{aligned} \tag{5}$$

**Theorem 2.2** ([31])**.** *Assume that $h$ is a Griffiths positive smooth Hermitian metric on $E$. Take $Y = \mathbb{P}(E^*)$ and*

$$(F, h_F) = (L_E^{m+r}\pi^* \det E^*, h_L^{m+r} \otimes \pi^* \det E^*),$$

*Then the $L^2$-metric induced by $h_F$ on*

$$\pi_*(K_{Y/X} \otimes F) = S^m E$$

*is exactly a constant multiple of $S^m h$.*

We will establish the validity of the above theorem for a Griffiths semi-positive singular Hermitian metric $h$ (refer to Section 3.1).

Finally, let us briefly introduce the results of the $L^2$ metric on direct image sheaves in the singular setting. For a projective fibration between compact complex manifold, Berndtsson–Păun in [3] showed the Griffiths semi-positivity of the $L^2$-metric on the direct image bundle. By using the geometric meaning of the optimal $L^2$ extension theorem found by Guan–Zhou in [19, 21], Hacon–Popa–Schnell in [23] generalized Berndtsson–Păun's result to encompass the direct image sheaf. Zhou–Zhu in [50] extended their work to Kähler case. Recently, Zou [53] and Watanabe [46] improved the positivity of such canonical singular metrics. For more insights into these results, we refer the readers to [32, 44, 39, 11, etc].

2.3. **Some results used in the proof.** In this subsection, we review several classical results, which are used in the proof of main theorems and applications.

**Proposition 2.3** ([3, 40, 39])**.** *If $(E, h) \geq_{\mathrm{Grif}} 0$, then locally there exists a family of Griffiths positive smooth Hermitian metrics $h_v$ such that $h_v$ increasingly converges to $h$.*

In fact, the approximating sequence is obtained through the technique of convolution with an approximate identity. Hence one immediately yields the following proposition.

**Proposition 2.4** ([39])**.** *Let $(E, h) \geq_{\mathrm{Grif}} 0$, then*

(i) $(Q, h_Q) \geq_{\mathrm{Grif}} 0$*, where $(Q, h_Q)$ is a quotient of $(E, h)$,*

(ii) $(S^m E, S^m h) \geq_{\mathrm{Grif}} 0$ *for $m \geq 1$,*

(iii) $(\Lambda^m E, \Lambda^m h) \geq_{\mathrm{Grif}} 0$*, especially* $(\det E, \det h) \geq_{\mathrm{Grif}} 0$ *for $m \geq 1$,*

(iv) $(\pi^* E, \pi^* h) \geq_{\mathrm{Grif}} 0$*, where $\pi : Y \to X$ is a holomorphic morphism.*

**Theorem 2.5** (Rückert's Nullstellensatz, [17, Section 3.1.2])**.** *Let $\mathcal{F}$ be a coherent analytic sheaf on a complex space $X$, and let $f \in \mathcal{O}_X(X)$ be a holomorphic function vanishing on* Supp $\mathcal{F}$*. Then, for each point $x \in X$, there exist an open neighborhood $U$ of $x$ in $X$ and a positive integer $t$ such that $f^t \mathcal{F}_U = 0$.*

**Theorem 2.6** (Remmert's proper mapping theorem, [17, Section 10.6.1])**.** *For any proper holomorphic map $f : X \to Y$ between complex spaces $X$, $Y$, the image set $f(X)$ is an analytic set in $Y$.*

**Theorem 2.7** (Grauert's upper semicontinuity theorem, [17, Section 10.5.5])**.** *Let $X$, $Y$ be complex manifolds. Assume $f : X \to Y$ is a proper submersion. Let $V$ be a holomorphic vector bundle on $X$.*

(i) *Then the set $A_{i,d} := \{y \in Y \mid \dim_{\mathbb{C}} H^i(X_y, V_y) \geq d\}$ is an analytic subset of $Y$, where $X_y := f^{-1}(y)$ and $V_y := V|_{X_y}$.*

(ii) *If $\dim_{\mathbb{C}} H^i(X_y, V_y)$ is independent of $y \in Y$, then all sheaves $R^i f_*(V)$ are locally free and all maps $(R^i f_*)_y : R^i f_*(V)/\mathfrak{m}_y R^i f_*(V) \to H^i(X_y, V_y)$ are isomorphisms.*

We revisit Leray's isomorphism theorem and Bott's vanishing theorem.

**Theorem 2.8** (Leray, [26, Section 3.5, Theorem 5.5])**.** *Let $X$ and $Y$ be topological spaces and $\pi : X \to Y$ be a proper map. Given a sheaf $\mathcal{F}$ of abelian groups over $X$, $R^q\pi_*\mathcal{F}$ is the higher direct image sheaf over $Y$, defined as the sheafification of presheaf*

$$U \subset Y \mapsto H^q(\pi^{-1}(U), \mathcal{F}).$$

*If for some fixed $p$,*

$$R^q\pi_*\mathcal{F} = 0, \quad \text{for all } q \neq p.$$

*Then for any $i$, there is a natural isomorphism*

$$H^i(X, \mathcal{F}) \simeq H^{i-p}(Y, R^p\pi_*\mathcal{F}).$$

**Theorem 2.9** (Bott, [26, Section 3.4, Theorem 4.10])**.** *For $p, q \geq 0$ and $k \in \mathbb{Z}$, then*

$$H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(k)) = 0$$

*with the following exceptions:*

(i) *$p = q$ and $k = 0$,*

(ii) *$q = 0$ and $k > p$,*

(iii) *$q = n$ and $k < p - n$.*

Now recall the definition of numerical dimension of singular metrics and the Kawamata–Viehweg–Nadel type vanishing theorem in [7].

Let $(M, h_M) \geq_{\text{Grif}} 0$ be a holomorphic line bundle over a compact Hermitian manifold $(X, \omega)$. A sequence $\{h_j\}$ is called a *quasi-equisingular approximation* of $h$ if it satisfies the following conditions:

(i) $-\log h_j$ converges to $-\log h$ in $L^1$ topology and

$$\sqrt{-1}\Theta_{(M,h_j)} \geq \sqrt{-1}\Theta_{(M,h_M)} - \delta_j\omega$$

for some positive number $\delta_j$ with $\delta_j \to 0$ as $j \to +\infty$;

(ii) all $-\log h_j$ have analytic singularities and $-\log h_{j+1} \leq -\log h_j + O(1)$;

(iii) for any $\delta > 0$ and $m \in \mathbb{N}$, there exists $j_0 = j(\delta, m) \in \mathbb{N}$ such that $\mathcal{I}(h_j^{m(1+\delta)}) \subseteq \mathcal{I}(h_M^m)$ for any $j \geq j_0$.

It is well-known that the existence of the quasi-equisingular sequence is ensured by Demailly's approximation theorem.

For any closed smooth semi-positive $(n-q, n-q)$ form $u$, one defines

$$\int_X \left(\sqrt{-1}\Theta_{(M,h_M)}\right)^q \wedge u := \limsup_{k\to\infty} \int_X \left(\sqrt{-1}\Theta_{(M,h_j)}\right)^q_{ac} \wedge u, \tag{6}$$

where $(\sqrt{-1}\Theta_{(M,h_j)})_{ac}$ denotes the absolutely continuous part of the current $\sqrt{-1}\Theta_{(M,h_j)}$.

In [7], when $(X, \omega)$ is a compact Kähler manifold, Cao proved that the limsup is independent of the choice of quasi-equisingular approximation and is a limit.

**Definition 2.10** ([7])**.** The numerical dimension of $(M, h_M)$ is defined as

$$\mathrm{nd}(M, h_M) := \max\{q \in \mathbb{N} \mid \left(\sqrt{-1}\Theta_{(M,h_M)}\right)^q \neq 0\}.$$

One has the following vanishing theorem for pseudoeffective line bundles.

**Theorem 2.11** ([7, 20])**.** *Let $(M, h_M) \geq_{\mathrm{Grif}} 0$ be a holomorphic line bundle on a compact Kähler manifold $(X, \omega)$. Then*

$$H^q(X, K_X \otimes M \otimes \mathcal{I}(h_M)) = 0, \quad \textit{for all } q \geq n - \mathrm{nd}(M, h_M) + 1.$$

**Definition 2.12.** A holomorphic morphism $\pi : Y \to X$ is called a *Kähler modification* of $X$ if $\pi$ is proper, and $Y$ is a Kähler manifold and there exists a nowhere dense closed analytic subset $B \subseteq Y$ such that the restriction $\pi|_{Y\backslash B} : Y\backslash B \to X\backslash\pi(B)$ is an isomorphism.

**Lemma 2.13.** *Let $(M, h_M) \geq_{\mathrm{Grif}} 0$ be a holomorphic line bundle on a compact complex manifold $X$ with analytic singularties. Suppose*

$$\int_X (\sqrt{-1}\Theta_{(M,h_M)})^{\dim X} > 0$$

*in the sense of* (6)*. Then $X$ admits a Kähler modification.*

*Proof.* By (6) and Bonavero's singular Morse inequality (see [35, Theorem 2.2.30]), the line bundle $M$ is big. Thus, $X$ is a Moishezen manifold, i.e., there exists a projective algebraic variety $Y$ with $\dim Y = \dim X$ and a bimeromorphic map $\mu : X \dashrightarrow Y$. By Hironaka's desingularization theorem, there exists a Kähler modification of $X$. □

Additionally, if one wishes to weaken the condition of the underlying manifolds, then we have Meng–Zhou's vanishing Theorem for holomorphically convex Kähler manifolds.

**Theorem 2.14** ([36])**.** *Let $X$ be a holomorphically convex Kähler manifold and $(L, h_L) \geq_{\mathrm{Grif}} 0$ be a holomorphic vector bundle. Suppose $h_L$ is smooth outside an analytic subset $Z$, satisfying $(\sqrt{-1}\Theta_{(L,h_L)})^\ell(x_0) \neq 0$ for some $x_0 \in X\backslash Z$. Then, for any $q \geq n - \ell + 1$, we have*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)) = 0.$$

The following lemma explains the factorial property of submodule sheaves under proper holomorphic modifications.

**Lemma 2.15** ([42])**.** *Let $\mu : \widetilde{X} \to X$ be a proper holomorphic modification and $(E, h) \geq_{\mathrm{Grif}} 0$ be a holomorphic vector bundle on a complex manifold $X$. Then*

$$\mu_*(K_{\widetilde{X}} \otimes \mu^*\mathcal{E}(\mu^*h)) = K_X \otimes \mathcal{E}(h),$$

*where $\mu^*\mathcal{E}(\mu^*h)$ is the multiplier submodule sheaf associated to $(\mu^*E, \mu^*h)$.*

*Proof.* By definition, there exists an analytic subset $A \subsetneq X$ such that the map

$$\mu : \widetilde{X}\backslash\mu^{-1}(A) \to X\backslash A$$

is a holomorphic bijection.

For any open subset $U \subseteq X$, $f \in H^0(U, K_X \otimes \mathcal{E}(h))$ is a $E$-valued holomorphic $(n, 0)$-form on $U$ with $|f|_h^2 \in L^1_{\mathrm{loc}}(U)$. Here, $\mu^*f$ denotes the pullback of the restriction on $U\backslash A$ of $f$.

Then, one has

$$\mu^*f \in H^0(\mu^{-1}(U)\backslash\mu^{-1}(A), K_{\widetilde{X}} \otimes \mu^*E)$$

and

$$\int_{\mu^{-1}(V)\backslash\mu^{-1}(A)} |\mu^* f|^2_{\mu^* h} = \int_{V\backslash A} |f|^2_h < +\infty$$

for any $V \Subset U$. Since $\mu^* h$ has a positive lower bound locally, $\mu^* f$ can be extended holomorphically across the analytic subset $\mu^{-1}(A)$, i.e.,

$$\mu^* f \in H^0(\mu^{-1}(U), K_{\widetilde{X}} \otimes \mu^*\mathcal{E}(\mu^* h)) = H^0(U, \mu_*(K_{\widetilde{X}} \otimes \mu^*\mathcal{E}(\mu^* h))).$$

Conversely, if $g \in H^0(\mu^{-1}(U), K_{\widetilde{X}} \otimes \mu^* E(\mu^* h))$, we denote by $(\mu^{-1})^* g$ the pullback of the restriction on $\mu^{-1}(U)\backslash\mu^{-1}(A)$ of $g$. By a similar argument, $(\mu^{-1})^* g$ can also be extended holomorphically across the analytic subset $A$, and

$$(\mu^{-1})^* g \in H^0(U, K_X \otimes \mathcal{E}(h)).$$

□

It is worth mentioning that the multiplier submodule sheaf itself does not possess the factorial property, motivating us to consider the isomorphism

$$\pi_*(K_{\mathbb{P}(E^*)} \otimes (L_E)^{m+r} \otimes \pi^* \det E^*) = K_X \otimes S^m E$$

instead of $\pi_*(L_E)^m = S^m E$.

The injectivity theorem for higher direct image sheaves, which has been widely studied in the last decades, is a powerful tool in complex geometry and algebraic geometry (see [27, 43, 14, 15, 34, etc]). A more general statement of injectivity theorem was established by Zhou–Zhu, as follows,

**Theorem 2.16** ([51]). *Let $X$ be a holomorphically convex Kähler manifold, $(F, h_F)$ and $(M, h_M)$ be two holomorphic line bundles with singular Hermitian metrics. Suppose*

$$\sqrt{-1}\Theta_{(F,h_F)} \geq 0,\ \sqrt{-1}\Theta_{(F,h_F)} \geq b\sqrt{-1}\Theta_{(M,h_M)}$$

*in the sense of currents for some positive constant $b$. Then for any global holomorphic section $s$ of $M$ satisfying*

$$\sup_{\Omega} |s|_{h_M} < +\infty$$

*for any $\Omega \Subset X$ and $q \geq 0$, the map*

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h_F)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F \otimes M \otimes \mathcal{I}(h_F \otimes h_M))$$

*is injective.*

As a corollary, one can obtain the torsion-freeness of higher direct image sheaves.

**Theorem 2.17** (Torsion-freeness, [34]). *Let $f$ be a proper holomorphic surjective morphism from a Kähler manifold $X$ to a complex analytic variety $Y$. Suppose $(F, h_F) \geq_{\mathrm{Grif}} 0$ be a holomorphic line bundle. Then, for any $k \geq 0$, the sheaf*

$$R^k f_*(K_X \otimes F \otimes \mathcal{I}(h_F))$$

*is torsion-free.*

*In particular, $R^k f_*(K_X \otimes F \otimes \mathcal{I}(h_F)) = 0$ for $k > \dim X - \dim Y$.*

*Proof.* Suppose there exists $s \in R^k f_*(K_X \otimes F \otimes \mathcal{I}(h_F))_y$ and $0 \neq t \in \mathcal{O}_{Y,y}$ for a given $y \in Y$, such that $t \cdot s = 0$. Let us consider a Stein space neighborhood $U$ of $y$ that is sufficiently small to define $s$ and $t$. By the assumption that $X$ is a holomorphically convex Kähler manifold, the preimage $U' := f^{-1}(U)$ is also a holomorphically convex Kähler manifold. Then $f^* t \in \mathcal{O}_X(U')$.

Applying

$$(M, h_M) = (U' \times \mathbb{C}, 1)$$

in Theorem 2.16, then $f^*t$ is a nonzero global section on $M$, satisfying the conditions of Theorem 2.16. Consequently, the map

$$H^k(U', K_X \otimes F \otimes \mathcal{I}(h_F)) \hookrightarrow H^k(U', K_X \otimes F \otimes \mathcal{I}(h_F)),$$

induced by tensoring $f^*t$, is an injection.

Since $t \cdot s = 0$, then $f^*t \cdot s = 0$, where the later $s$ is regarded as an element of $H^k(U', K_X \otimes F \otimes \mathcal{I}(h_F))$. Therefore, by Theorem 2.16, we conclude that $s = 0$.

This implies that

$$R^k f_*(K_X \otimes F \otimes \mathcal{I}(h_F))$$

is torsion-free, since $y$ is arbitrary. □

**Remark 2.18.** (i) If the torsion-free sheaf $R^k f_*(K_X \otimes F \otimes \mathcal{I}(h_F))$ is supported on a proper analytic subset, then according to Rückert's Nullstellensatz (Theorem 2.5), we obtain

$$R^k f_*(K_X \otimes F \otimes \mathcal{I}(h_F)) = 0.$$

(ii) We can also derive same results to those in Theorem 2.17 under a weaker condition: the existence of an open covering $\{U_\alpha\}$ of $Y$ such that $f^{-1}(U_\alpha)$ is Kähler, compared to the Kählerity of $X$.

It is evident from Lemma 2.15, Theorem 2.17, and Theorem 2.8 that the cohomology groups with multiplier ideal sheaves remain invariant under Kähler modifications.

**Lemma 2.19.** *Let $(F, h_F) \geq_{\mathrm{Grif}} 0$ be a holomorphic line bundle over a complex manifold $X$ and $\mu : \widetilde{X} \to X$ be a Kähler modification. Then for any $q \geq 0$,*

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h_F)) \simeq H^q(\widetilde{X}, K_{\widetilde{X}} \otimes \mu^* F \otimes \mathcal{I}(\mu^* h_F)).$$

## 3. Proof of the Main Theorem

In this section, we present a proof of main Theorem 1.3.

3.1. **Proof of Theorem 1.3-(i).** In light of Leray's theorem 2.8, our task is to demonstrate the isomorphism of sheaves (Proposition 3.1) and the vanishing of the higher direct image sheaves (Lemma 3.6).

Initially, there is a one-to-one corresponding between $\Gamma(U, S^m E)$ and $\Gamma(\pi^{-1}(U), L_E^m)$. Furthermore, we can extend the isomorphisms twisted with multiplier submodule sheaves.

**Proposition 3.1.** *Let $(E, h) \geq_{\mathrm{Grif}} 0$ be a holomorphic vector bundle on a complex manifold $X$. Then we have*

$$S^m \mathcal{E}(S^m h) = \pi_*(L_E^m \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)).$$

*Proof.* Firstly, we have

$$S^m E = \pi_*(K_{\mathbb{P}(E^*)/X} \otimes L_E^{m+r} \otimes \pi^* \det E^*).$$

Hence, our objective is to show the equivalence of integrability concerning the corresponding sections.

Since $(E, h)$ is Griffiths semi-positive and the question is local, by Proposition 2.3, we may assume that $E$ is trivial and there exists a family of Griffith positive smooth Hermitian metrics $\{h_\nu\}$ on $X$ increasingly converging to $h$. Then the metrics $\{h_{\nu,L}\}$ induced by $\{h_\nu\}$ are smooth positive and increasingly converge to $h_L$. Then by Theorem 2.2, the $L^2$-metric induced by $h_{\nu,L}^{m+r}$ is equal to a constant multiple of $S^m h_\nu \otimes \det h_\nu$.

Consider the setting with $Y := \mathbb{P}(E^*)$ and

$$(F, h_F) := (L_E^{m+r} \otimes \pi^* \det E^*, h_L^{m+r} \otimes \pi^* \det h^*).$$

For any $x \in X$ with $\det h(x) < +\infty$ and $u \in S^m E_x = H^0(Y_x, K_{Y_x} \otimes F|_{Y_x})$, the induced $L^2$-metric $h_{h_F}$ satisfies

$$\begin{aligned}
|u|^2_{h_{h_F}}(x) &:= \int_{Y_x} |u|^2_{h_F} \\
&= \int_{Y_x} |u|^2_{h_L^{m+r} \otimes \pi^* \det h^*} \\
&= \det h^*(x) \int_{Y_x} |u|^2_{h_L^{m+r}} \\
&= \det h^*(x) \lim_\nu \int_{Y_x} |u|^2_{h_{\nu,L}^{m+r}} \\
&= C_{m,r} \det h^*(x) \lim_\nu |u|^2_{S^m h_\nu}(x) \cdot \det h_\nu(x) \\
&= C_{m,r} |u|^2_{S^m h}(x), \qquad (7)
\end{aligned}$$

where $C_{m,r}$ is a constant only dependent on $m$ and $r$.

In summary, the $L^2$-metric induced by $h_L^{m+r} \otimes \pi^* \det h^*$ is well-defined and equals to a constant multiple of $S^m h$ almost everywhere. Moreover, for any suitably small open subset $U$ and $u \in H^0(U, S^m \mathcal{E}(S^m h))$, we conclude from the definition, (7) and Fubini's theorem that for each $U' \Subset U$,

$$\begin{aligned}
+\infty &> \int_{U'} |u|^2_{S^m h} d\lambda(x) \\
&= C_{m,r}^{-1} \int_{U'} \int_{Y_x} |u|^2_{h_F} d\lambda(x) \\
&= C_{m,r}^{-1} \int_{\pi^{-1}(U')} |u|^2_{h_F} \pi^* d\lambda(x),
\end{aligned}$$

where $d\lambda(x) = \sqrt{-1}^{n^2} dx \wedge d\overline{x}$ is a volume form on $U$. This implies that $u \wedge \pi^* dx \in H^0(\pi^{-1}(U), K_Y \otimes F)$ is $L^2$ integrable on $\pi^{-1}(U')$ with respect to $h_F$ and hence

$$u \in H^0(\pi^{-1}(U), K_{Y/X} \otimes F \otimes \mathcal{I}(h_F)).$$

Conversely, take $u \in H^0(\pi^{-1}(U), K_{Y/X} \otimes F \otimes \mathcal{I}(h_F))$, then for any $U' \Subset U$, it follows from the finite covering theorem, (7) and Fubini's theorem that $u$ is $L^2$ integrable on $U'$ with respect to $S^m h$. Hence $u \in H^0(U, S^m \mathcal{E}(S^m h))$.

Consequently, we establish that

$$(K_X \otimes S^m \mathcal{E}(S^m h))(U) \simeq \pi_*(K_{\mathbb{P}(E^*)} \otimes L_E^{m+r} \otimes \pi^* \det E^* \otimes \mathcal{I}(h_L^{m+r} \cdot \pi^* \det h^*))(U).$$

This concludes the proof of Proposition 3.1.

□

**Remark 3.2.** From the proof, we can strengthen Proposition 3.1 to the following result:

Let $(E, h) \geq_{\mathrm{Grif}} 0$ be a holomorphic vector bundle and $M$ be a holomorphic line bundle with a singular Hermitian metric $h_M$ on a Hermitian manifold $(X, \omega)$. Assume that

$$\sqrt{-1}\Theta_{(M,h_M)} \geq -C\omega$$

for some constant $C$. Then we have

$$S^m\mathcal{E}\otimes\mathcal{M}(S^m h\otimes h_M)=\pi_*(L_E^m\otimes\pi^*M\otimes\mathcal{I}(h_L^{m+r}\otimes\pi^*\det h^*\otimes\pi^*h_M)),$$

where $S^m\mathcal{E}\otimes\mathcal{M}(S^m h\otimes h_M)$ is the multiplier submodule sheaf associated to $(S^mE\otimes M, S^mh\otimes h_M)$.

Recall that Deng–Ning–Wang–Zhou [11] introduced a new notion "optimal $L^2$-estimate condition" and gave a characterization of Nakano positivity for smooth Hermitian metrics on holomorphic vector bundles via this notion. Subsequently, Inayama [24] provided a definition of Nakano positivity of singular Hermitian metrics on holomorphic vector bundles based on this characterization.

**Definition 3.3** ([11, 24])**.** Let $(E,h)$ be a Griffiths semi-postive singular Hermitian holomorphic vector bundle. Then $h$ is called Nakano semi-positive, denoted by $(E,h)\geq_{\mathrm{Nak}}0$, if $h$ is $L^2$ optimal (optimal $L^2$-estimate condition):

for any Stein coordinate $U$ such that $E|_U$ is trivial, any Kähler form $\omega_U$ on $U$, any $\psi\in\mathrm{Spsh}(U)\cap C^\infty(U)$ and any $\overline{\partial}$-closed $f\in L^2_{(n,1)}(U,\omega_U,E|_U,he^{-\psi})$, there exists $u\in L^2_{(n,0)}(U,\omega_U,E|_U,he^{-\psi})$ such that $\overline{\partial}u=f$ and

$$\int_U |u|^2_{\omega_U,h}e^{-\psi}dV_{\omega_U}\leq\int_U\langle B_\psi^{-1}f,f\rangle_{\omega_U,h}e^{-\psi}dV_{\omega_U},\tag{8}$$

provided that the right hand side is finite, where $B_\psi=[\sqrt{-1}\partial\overline{\partial}\psi\otimes\mathrm{Id}_E,\Lambda_{\omega_U}]$.

Essentially following the idea of [11, Theorem 4.3] together with (7), it is deduced that strongly Nakano positivity implies Nakano positivity.

**Proposition 3.4.** *Let $X$ be a complex manifold and $(E,h)\geq_{\mathrm{SNak}}0$. Then $h$ is a Nakano semi-positive singular Hermitian metric.*

*Proof.* Let us denote the holomorphic line bundle

$$(F,h_F):=(L_E^{r+1}\otimes\pi^*\det E^*, h_L^{r+1}\otimes\pi^*\det h^*).$$

The condition $(E,h)\geq_{\mathrm{SNak}}0$ implies $(F,h_F)\geq_{\mathrm{Grif}}0$. By formula (1), we have $E=\pi_*(K_{\mathbb{P}(E^*)/X}\otimes F)$.

It suffices to show that $(E,h)$ is $L^2$ optimal. Consider any Stein coordinate $(U,z)$ such that $E|_U$ is trivial and any Kähler metric $\omega_U$ on $U$. Let $f$ be a $\overline{\partial}$-closed smooth $(n,1)$-form with values in $E$, and $\psi$ be any smooth strictly plurisubharmonic function on $U$. We express

$$f(z)=dz\wedge(f_1(z)d\overline{z}_1+\cdots+f_n(z)d\overline{z}_n),$$

with $f_i(z)\in E_z=H^0(\mathbb{P}(E^*)_z,K_{\mathbb{P}(E^*)_z}\otimes F)$. One can identify $f$ as a smooth $(n+r-1,1)$-form $\widetilde{f}(z,w):=dz\wedge(f_1(z,w)d\overline{z}_1+\cdots+f_n(z,w)d\overline{z}_n)$ on $\pi^{-1}(U)$, where $f_i(z,w)$ represents holomorphic sections of $K_{\mathbb{P}(E^*)}\otimes F|_{\pi^{-1}(z)}$.

We observe the following:

(a) $\overline{\partial}_w f_i(z,w)=0$ for any fixed $z\in U$, as $f_i(z,w)$ are holomorphic sections of $K_{\mathbb{P}(E^*)_z}\otimes F|_{\pi^{-1}(z)}$.

(b) $\overline{\partial}_z f=0$, since $f$ is a $\overline{\partial}$-closed form.

It follows that $\widetilde{f}$ is a smooth $\overline{\partial}$-closed $(n+r-1,1)$-form on $U\times\mathbb{P}^{r-1}$ with values in $F$.

Since $f$ is $E$-valued $(n,1)$-form, $\langle B_\psi^{-1}f,f\rangle_{\omega_U,h}dV_{\omega_U}$ is independent of the choice of $\omega_U$ and thus

$$\langle B_\psi^{-1}f,f\rangle_{\omega_U,h}dV_{\omega_U}=\sum_{j,k=1}^n\psi^{jk}\langle f_j,f_k\rangle_h\sqrt{-1}^{n^2}dz\wedge d\overline{z},$$

where $(\psi^{jk}) = (\frac{\partial^2\psi}{\partial z_j \partial \bar{z}_k})^{-1}$.

And since each $f_j|_{\mathbb{P}(E^*)_z}$ is holomorphic $F|_{\mathbb{P}(E^*)_z}$-valued $(r-1,0)$-form, then

$$\int_{\mathbb{P}(E^*)_z} \langle f_j, f_k \rangle_{\omega_{\pi^{-1}(U)}|_{\mathbb{P}(E^*)_z}, h_F} dV_{\omega_{\pi^{-1}(U)}|_{\mathbb{P}(E^*)_z}}$$

does not depend on the choice of $\omega_{\pi^{-1}(U)}$. Hence for any fixed $z \in U$ with $\det h(z) < +\infty$, we have

$$\begin{aligned}
&\int_{\mathbb{P}(E^*)_z} \langle B^{-1}_{\pi^*\psi} \widetilde{f}, \widetilde{f} \rangle_{\omega_{\pi^{-1}(U)}|_{\mathbb{P}(E^*)_z}, h_F} dV_{\omega_{\pi^{-1}(U)}|_{\mathbb{P}(E^*)_z}} \\
&= \sum_{j,k=1}^{n} \psi^{jk}(z) \left( \int_{\mathbb{P}(E^*)_z} \langle f_j, f_k \rangle_{\omega_{\pi^{-1}(U)}|_{\mathbb{P}(E^*)_z}, h_F} dV_{\omega_{\pi^{-1}(U)}|_{\mathbb{P}(E^*)_z}} \right) \sqrt{-1}^{n^2} dz \wedge d\overline{z} \\
&= \sum_{j,k=1}^{n} \psi^{jk}(z) \langle f_j, f_k \rangle_h \sqrt{-1}^{n^2} dz \wedge d\overline{z},
\end{aligned}$$

where the last equality is due to the formula (7).

Then by the Fubini–Tonelli theorem, we obtain that

$$\int_{\pi^{-1}(U)} \langle B^{-1}_{\pi^*\psi} \widetilde{f}, \widetilde{f} \rangle_{\omega_{\pi^{-1}(U)}, h_F} e^{-\pi^*\psi} dV_{\omega_{\pi^{-1}(U)}} = \int_U \langle B^{-1}_{\psi} f, f \rangle_{\omega_U, h} e^{-\psi} dV_{\omega_U} < +\infty.$$

Notice that $h_F$ is semi-positive on $F$, then there exists an $(n+r-1,0)$-form $\widetilde{u}$ with $L^2$ coefficients such that $\overline{\partial}\widetilde{u} = \widetilde{f}$ and

$$\begin{aligned}
&\int_{\pi^{-1}(U)} |\widetilde{u}|^2_{\omega_{\pi^{-1}(U)}, h_F} e^{-\pi^*\psi} dV_{\omega_{\pi^{-1}(U)}} \\
&\leq \int_{\pi^{-1}(U)} \langle B^{-1}_{\pi^*\psi} \widetilde{f}, \widetilde{f} \rangle_{\omega_{\pi^{-1}(U)}, h_F} e^{-\pi^*\psi} dV_{\omega_{\pi^{-1}(U)}} \\
&= \int_U \langle B^{-1}_{\psi} f, f \rangle_{\omega_U, h} e^{-\psi} dV_{\omega_U}.
\end{aligned}$$

Additionally, due to the weak regularity of $\overline{\partial}$ on $(n+r-1,0)$-forms, we can take $\widetilde{u}$ to be smooth. It's observed that $\overline{\partial}\widetilde{u}|_{P(E^*)_z} = 0$ for any fixed $z \in U$, due to $\overline{\partial}\widetilde{u} = \widetilde{f}$ and $\overline{\partial}_w f_i(z,w) = 0$. This means that $\widetilde{u}_z := \widetilde{u}(z,\cdot) \in E_z$. Consequently, we may view $\widetilde{u}$ as a section $u$ of $E$. Then $\overline{\partial}u = f$ and

$$\begin{aligned}
\int_U |u|^2_{\omega_U, h} e^{-\psi} dV_{\omega_U} &= \int_{\pi^{-1}(U)} |\widetilde{u}|^2_{\omega_{\pi^{-1}(U)}, h_F} e^{-\pi^*\psi} dV_{\omega_{\pi^{-1}(U)}} \\
&\leq \int_U \langle B^{-1}_{\psi} f, f \rangle_{\omega_U, h} e^{-\psi} dV_{\omega_U}.
\end{aligned}$$

□

**Remark 3.5.** Similarly, we can show that $(E,h) \geq_{\mathrm{SNak}} 0$ implies that $(S^m E, S^m h) \geq_{\mathrm{Nak}} 0$ for any $m \geq 1$.

We now establish the vanishing of higher direct image sheaves, thereby concluding the proof of Theorem 1.3-(i).

**Lemma 3.6.** *Assume $X$ is a complex manifold. Let $(E,h) \geq_{\mathrm{SNak}} 0$ be a holomorphic vector bundle of rank $r$ on $X$. Then for any $k, m > 0$, we have*

$$R^k \pi_* \left( L^m_E \otimes \mathcal{I}(h^{m+r}_L \otimes \pi^* \det h^*) \right) = 0.$$

*Proof.* To begin, we rephrase the expression:

$$L_E^m \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*) = K_{\mathbb{P}(E^*)/X} \otimes L_E^{r+m} \otimes \pi^* \det E^* \otimes \mathcal{I}(h_L^{r+m} \otimes \pi^* \det h^*).$$

Since $h_L^{m+r} \otimes \pi^* \det h^*$ is Griffiths semi-positive, then the sheaf $\mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)$ is coherent. This implies that the set

$$A := \operatorname{Supp} \mathcal{O}_{\mathbb{P}(E^*)} \big/ \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)$$

is an analytic subset of $\mathbb{P}(E^*)$. It follows from Remmert's proper mapping theorem (see Theorem 2.6) that $\pi(A)$ is an analytic subset of $X$.

Consider a Stein open set $U \subset X$ such that $E|_U$ is trivial. For any $x \in \{\det h < +\infty\}$, the restricted metric $h_L|_{\pi^{-1}(x)}$ is smooth Griffiths positive. and so is $(h_L^{m+r} \otimes \pi^* \det h^*)|_{\pi^{-1}(x)}$. Since the fiber $\mathbb{P}^{r-1}$ is quasi-Stein, then by Ohsawa–Takegoshi $L^2$-extension, we know that such fiber $\pi^{-1}(x)$ does not intersect with $A$. Thus, $\pi(A) \subsetneq X$.

Consider the projection map

$$\pi : (U \setminus \pi(A)) \times \mathbb{P}^{r-1} \to U \setminus \pi(A).$$

Since $\mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)$ is trivial over $(U \setminus \pi(A)) \times \mathbb{P}^{r-1}$, then by Grauert's upper semicontinuity theorem (see Theorem 2.7-(i)), we deduce that

$$x \mapsto \dim H^k(\pi^{-1}(x), (K_{\mathbb{P}(E^*)/X} \otimes L_E^{r+m} \otimes \pi^* \det E^*)|_{\pi^{-1}(x)})$$

is a locally constant function on a non-empty Zariski open set $U' \subset U \setminus \pi(A)$. Furthermore, thanks to Theorem 2.7-(ii), for any $x \in U'$, the following isomorphism holds:

$$\begin{aligned} & \frac{(R^k \pi_*(K_{\mathbb{P}(E^*)/X} \otimes L_E^{r+m} \otimes \pi^* \det E^*))_x}{\mathfrak{m}_x (R^k \pi_*(K_{\mathbb{P}(E^*)/X} \otimes L_E^{r+m} \otimes \pi^* \det E^*))_x} \\ \simeq \; & H^k(\pi^{-1}(x), (K_{\mathbb{P}(E^*)/X} \otimes L_E^{r+m} \otimes \pi^* \det E^*)|_{\pi^{-1}(x)}). \end{aligned}$$

Additionally, we observe the following:

(i) $\pi^{-1}(x) = \mathbb{P}^{r-1}$ is compact Kähler,

(ii) $(K_{\mathbb{P}(E^*)/X} \otimes L_E^{r+m} \otimes \pi^* \det E^*)|_{\pi^{-1}(x)} = K_{\pi^{-1}(x)} \otimes (L_E^{r+m} \otimes \pi^* \det E^*)|_{\pi^{-1}(x)}$,

(iii) for $x \in \{\det h < +\infty\} \cap U'$, $(h_L^{r+m} \otimes \pi^* \det h^*)|_{\pi^{-1}(x)}$ is a smooth Griffiths positive metric on $(L_E^{r+m} \otimes \pi^* \det E^*)|_{\pi^{-1}(x)}$.

Consequently, by the Kodaira vanishing theorem, it follows that

$$H^k(\pi^{-1}(x), (K_{\mathbb{P}(E^*)/X} \otimes L_E^{r+m} \otimes \pi^* \det E^*)|_{\pi^{-1}(x)}) = 0,$$

for any $x \in \{\det h < +\infty\} \cap U'$ and any $k > 0$.

Hence by the Nakayama Lemma, for $x \in \{\det h < +\infty\} \cap U'$ and $k > 0$, we have

$$(R^k \pi_*(K_{\mathbb{P}(E^*)/X} \otimes L_E^{r+m} \otimes \pi^* \det E^*))_x = 0. \tag{9}$$

Therefore, for $x \in \{\det h < +\infty\} \cap U'$ and $k > 0$, we obtain

$$(R^k \pi_*(L_E^m \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)))_x = 0. \tag{10}$$

Since $R^k \pi_*(L_E^m \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*))$ is coherent, then it is supported on a proper analytic subset of $U$.

Additionally, combining with Theorem 2.17 and Remark 2.18-(ii), it can be concluded that

$$R^k \pi_*(L_E^m \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*))$$

is torsion-free. Thus, according to Remark 2.18-(i), we infer that

$$R^k \pi_*(L_E^m \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^*)) = 0.$$

□

**Remark 3.7.** The proof allows us to strengthen Lemma 3.6 to the following assertion:

Let $(E, h)$ be a holomorphic vector bundle and $(M, h_M)$ be a holomorphic line bundle on a Hermitian manifold $(X, \omega)$. Suppose that

$$\sqrt{-1}\Theta_{(L_E^{m+r}, h_L^{m+r})} - \pi^*\sqrt{-1}\Theta_{(\det E, \det h)} + \pi^*\sqrt{-1}\Theta_{(M, h_M)} \geq -C\pi^*\omega$$

for some constant $C$. Then, for any $k > 0$, we obtain

$$R^k\pi_*(L_E^m \otimes \pi^*M \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^* \otimes \pi^* h_M)) = 0.$$

Moreover, a more general statement than Theorem 1.3 can be obtained via Remark 3.2, Theorem 2.8 and Remark 3.7.

**Theorem 3.8.** *Let $(E, h) \geq_{\mathrm{Grif}} 0$ and $W$ be holomorphic vector bundles and $(M, h_M)$ be a holomorphic line bundle on a Hermitian manifold $(X, \omega)$. Assume that*

$$\sqrt{-1}\Theta_{(M, h_M)} \geq -C\omega$$

*and*

$$\sqrt{-1}\Theta_{(L_E^{m+r}, h_L^{m+r})} - \pi^*\sqrt{-1}\Theta_{(\det E, \det h)} + \pi^*\sqrt{-1}\Theta_{(M, h_M)} \geq -C\pi^*\omega$$

*for some positive constant $C$. Then for any $p, q \geq 0, m > 0$, we have*

$$\begin{aligned} &H^q(X, W \otimes S^m\mathcal{E} \otimes \mathcal{M}(S^m h \otimes h_M)) \\ \simeq &H^q(\mathbb{P}(E^*), \pi^*W \otimes L_E^m \otimes \pi^*M \otimes \mathcal{I}(h_L^{m+r} \otimes \pi^* \det h^* \otimes \pi^* h_M)). \end{aligned}$$

3.2. **Proof of Theorem 1.3-(ii).** To establish the second result of Theorem 1.3, we consider $W = \Omega_X^p$ and $m = 1$ in Theorem 1.3-(i), which yields an isomorphism of cohomologies:

$$H^q(X, \Omega_X^p \otimes \mathcal{E}(h)) \simeq H^q(\mathbb{P}(E^*), \pi^*\Omega_X^p \otimes L_E \otimes \mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)). \tag{11}$$

Hence, the rest is to construct an isomorphism between the cohomologies of $\pi^*\Omega_X^p \otimes L_E \otimes \mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)$ and $\Omega^p_{\mathbb{P}(E^*)} \otimes L_E \otimes \mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)$.

For convenience, we denote these sheaves by

$$\begin{aligned} \mathcal{F}^p &:= \pi^*\Omega_X^p \otimes L_E \otimes \mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*) \\ \mathcal{G}^p &:= \Omega^p_{\mathbb{P}(E^*)} \otimes L_E \otimes \mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*). \end{aligned}$$

Our goal now is to find an open covering such that $\mathcal{F}^p$ and $\mathcal{G}^p$ have the same sections on these open sets, and their higher cohomologies both vanish.

**Lemma 3.9.** *Let $U \subset X$ be a Stein open set such that $E$ is locally trivial on $U$. Then the inclusion $\mathcal{F}^p \subset \mathcal{G}^p$ induces an isomorphism*

$$H^0(\pi^{-1}(U), \mathcal{F}^p) \simeq H^0(\pi^{-1}(U), \mathcal{G}^p); \tag{12}$$

*Proof.* Since the multiplier ideal sheaves on both sides of (12) are identical, it suffices to show the isomorphism of bundles on both sides of (12).

For any $x \in U$, we have $L_E|_{\pi^{-1}(x)} = \mathcal{O}_{\mathbb{P}^{r-1}}(1)$ and $(\pi^*\Lambda^p T^*X)|_{\pi^{-1}(x)}$ forms a product bundle. From the decomposition

$$(\Lambda^p T^*\mathbb{P}(E^*))|_{\pi^{-1}(x)} = \bigoplus_{s+t=p} (\pi^*\Lambda^s T_x^*X \otimes \Lambda^t T^*\mathbb{P}(E_x^*)),$$

we deduce that

$$\begin{aligned} H^0(\pi^{-1}(x), \Omega^p_{\mathbb{P}(E^*)} \otimes L_E) = &\bigoplus_{s+t=p} (\Lambda^s T_x^*X) \otimes H^0(\pi^{-1}(x), \Omega^t_{\mathbb{P}(E_x^*)} \otimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)) \\ = &(\Lambda^p T_x^*X) \otimes H^0(\pi^{-1}(x), L_E|_{\pi^{-1}(x)}), \end{aligned}$$

where the second equality is a consequence of Bott's vanishing theorem (see Theorem 2.9)

$$H^0(\mathbb{P}^{r-1}, \Omega^t_{\mathbb{P}^{r-1}} \otimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)) = 0, \text{ for any } t > 0.$$

□

**Lemma 3.10.** *As above notation, then for any $p \geq 0$, $q > 0$,*

(i) $H^q(\pi^{-1}(U), \mathcal{F}^p) = 0$ ,
(ii) $H^q(\pi^{-1}(U), \mathcal{G}^p) = 0$.

*Proof.* (i) As in the proof of Lemma 3.6, we obtain that

$$\begin{aligned}\frac{(R^q\pi_*\mathcal{F}^p)_x}{\mathfrak{m}_x(R^q\pi_*\mathcal{F}^p)_x} \simeq & H^q(\pi^{-1}(x), \mathcal{F}^p|_{\pi^{-1}(x)}) \\ \simeq & H^q(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(1)) = 0,\end{aligned}$$

for $x \in U' \cap \{\det h < +\infty\}$ where $U' \subseteq U$ is a Zariski open set in $U$. Consequently, we obtain

$$R^q\pi_*\mathcal{F}^p = 0.$$

Hence, $H^q(\pi^{-1}(U), \mathcal{F}^p) = 0$.

(ii) Similarly, for $x \in U' \cap \{\det h < +\infty\}$, where $U' \subseteq U$ is a Zariski open set in $U$, we derive that

$$\frac{(R^q\pi_*\mathcal{G}^p)_x}{\mathfrak{m}_x(R^q\pi_*\mathcal{G}^p)_x} \simeq H^q(\mathbb{P}^{r-1}, (\oplus_{0\leq t\leq p}\Omega^t_{\mathbb{P}^{r-1}}) \otimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)) = 0,$$

by Bott's vanishing theorem on $\mathbb{P}^{r-1}$ (see Theorem 2.9). □

We now complete the proof of Theorem 1.3-(ii). Take a Stein open covering $\{U_j\}$ of $X$ such that $E|_{U_j}$ is trivial. Denoted by $V_j := \pi^{-1}(U_j)$, then $\{V_j\}$ forms an open covering of $\mathbb{P}(E^*)$. By Lemma 3.10, the covering $\{V_j\}$ is a Leray covering for sheaves $\mathcal{F}^p$ and $\mathcal{G}^p$ such that the sheaf cohomologies $H^*(\mathbb{P}(E^*), \mathcal{F}^p)$ and $H^*(\mathbb{P}(E^*), \mathcal{G}^p)$ can be computed using the open covering $\{V_j\}$. Together with Lemma 3.9, we arrive at

$$H^q(\mathbb{P}(E^*), \mathcal{F}^p) \simeq H^q(\mathbb{P}(E^*), \mathcal{G}^p). \tag{13}$$

By combining (11) and (13), we derive the result.

## 4. Applications

In this section, we give some applications of Theorem 1.3 and Theorem 1.6.

4.1. **Coherence and Strong openness.** Suppose $(E, h) \geq_{\text{Grif}} 0$ holds, Inayama in [25] raised the question whether the submodule sheaf $\mathcal{E}(h)$ is coherent. By Proposition 3.1 and Grauert's direct image theorem, it suffices to show the coherence of the ideal sheaf $\mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)$. Moreover, building upon the idea of Zou [54], one can prove that $\mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)$ is coherent when $\det h$ has analytic singularities. In summary, we obtain the following result.

**Proposition 4.1.** *Let $(E, h) \geq_{\text{Grif}} 0$.*

(i) *If $\mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)$ is coherent, then $\mathcal{E}(h)$ is coherent.*
(ii) *If $\det h$ has analytic singularities, then $\mathcal{I}(h_L^{1+r} \otimes \pi^* \det h^*)$ is coherent.*

The subsequent result is the strong openness of the multiplier submodule sheaf associated to strongly Nakano semi-positive Hermitian vector bundles.

**Proposition 4.2.** *Let $(E, h) \geq_{\mathrm{SNak}} 0$ be a holomorphic vector bundle on a complex manifold $X$. Then for any $m \geq 0$, $S^m\mathcal{E}(S^m h)$ satisfies the strong openness property, which means that one has*

$$\bigcup_j S^m\mathcal{E}(S^m h_j) = S^m\mathcal{E}(S^m h)$$

*for a decreasing sequence $\{h_j\}$ of Griffiths semi-positive metrics converging to $h$.*

*Proof.* As the result is local, we may assume that $E$ is a trivial vector bundle on the unit polydisk $\Delta^n$ and $h \geq \mathrm{Id}_E$.

Initially, we consider the the special case that $h_\varepsilon = h(\det h)^\varepsilon$, regarded as a new metric on $E$ that converges decreasingly to $h$ as $\varepsilon$ tends to 0. Since

$$h_{\varepsilon,L}^{m+r} \otimes \pi^* \det h_\varepsilon^* = h_L^{m+r} \otimes \pi^* \det h^* \cdot (\det h)^{m\varepsilon}$$

is also Griffiths semi-positive and decreasingly converges to $h_L^{m+r} \otimes \pi^* \det h^*$, it follows from Theorem 1.3 and the strong openness conjecture proved by Guan–Zhou [20] that

$$\bigcup_{\varepsilon>0} S^m\mathcal{E}(S^m h_\varepsilon) = S^m\mathcal{E}(S^m h).$$

For the general case, one can check that $S^m h \cdot (\det h)^{-m}$ is locally bounded from above and decreases with respect to $h$. In other words, if $h' \geq h$, then

$$S^m h' \cdot (\det h')^{-m} \leq S^m h \cdot (\det h)^{-m}.$$

Take a holomorphic section $F$ of $S^m E$ such that $|F|^2_{S^m h}$ is integrable on $\Delta^n$ and a compact subset $K \subset \Delta^n$. Denote $-\log\det h_j$ and $-\log\det h$ by $\varphi_j$ and $\varphi$ respectively, which are plurisubharmonic functions. This leads to

$$\int_K |F|^2_{S^m h_j} \leq \int_K |F|^2_{S^m h} e^{m\varphi - m\varphi_j}.$$

We will show that $|F|^2_{S^m h} \in L^{1+\varepsilon}(K)$ for some sufficiently small $\varepsilon > 0$ and for any fixed $q > 1$, $e^{m\varphi - m\varphi_j} \in L^q(K)$ for $j >> 1$. These facts imply $|F|^2_{S^m h_j} \in L^1(K)$ for $j >> 1$ by Hölder's inequality.

On the one hand, We may assume $|F|^2_{S^m h} \cdot (\det h)^{-m} < C$ on $K$ due to the local boundedness of $S^m h \cdot (\det h)^{-m}$, then

$$(|F|^2_{S^m h})^{1+\varepsilon} \leq C^\varepsilon |F|^2_{S^m h} (\det h)^{m\varepsilon},$$

where the right hand side is integrable on $K$ for some $\varepsilon > 0$ owing to the strong openness of the special case.

On the other hand, it follows from Demailly's approximation theorem [10, Theorem 14.2] that there are finite holomorphic functions $f_\alpha, \alpha = 1, \ldots, N$, on $\Delta^n$ with $\int_{\Delta^n} |f_\alpha|^2 e^{-qm\varphi} = 1$ and

$$qm\varphi \leq \log(\sum_\alpha |f_\alpha|^2) + O(1) \quad \text{on } K.$$

Then $e^{qm\varphi - qm\varphi_j} \leq C \sum_\alpha |f_\alpha|^2 e^{-qm\varphi_j}$ for some constant $C_K$ on $K$. As $qm\varphi_j \leq qm\varphi$ increasingly converges to $qm\varphi$, it follows that

$$\int_K e^{qm\varphi - qm\varphi_j} \leq C_K \sum_{\alpha=1}^N \int_K |f_\alpha|^2 e^{-qm\varphi_j} < +\infty$$

for $j$ large enough due to the strong openness conjecture proved by Guan–Zhou [20] .

□

**Remark 4.3.** In fact, since the strongly Nakano positivity implies Nakano positivity by Proposition 3.4, we can obtain strong openness of multiplier submodule sheaves directly from [33, Theorem 1.3].

4.2. **Direct sum and a counterexample.** In this subsection, we consider the direct sum of holomorphic vector bundles with direct sum metrics.

**Corollary 4.4.** *Let $(E_i, h_i) \geq_{\mathrm{SNak}} 0$ be holomorphic vector bundles of rank $r_i$ on a complex manifold $X$, for $i = 1, \dots, \ell$. If $W$ is a holomorphic vector bundle over $X$, then for $q \geq 0$, we have*

$$\begin{aligned} &H^q(X, W \otimes (\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell)(h_1 \oplus \cdots \oplus h_\ell)) \\ \simeq &H^q(\mathbb{P}(E_1^*), \pi_1^* W \otimes L_{E_1} \otimes \mathcal{I}(h_{L_{E_1}}^{1+r_1} \otimes \pi_1^* \det h_1^*)) \\ &\qquad \oplus \cdots \oplus H^q(\mathbb{P}(E_\ell^*), \pi_\ell^* W \otimes L_{E_\ell} \otimes \mathcal{I}(h_{L_{E_\ell}}^{1+r_\ell} \otimes \pi_\ell^* \det h_\ell^*)) \end{aligned}$$

*and*

$$\begin{aligned} &H^q(X, \Omega_X^p \otimes (\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell)(h_1 \oplus \cdots \oplus h_\ell)) \\ \simeq &H^q(\mathbb{P}(E_1^*), \Omega^p_{\mathbb{P}(E_1^*)} \otimes L_{E_1} \otimes \mathcal{I}(h_{L_{E_1}}^{1+r_1} \otimes \pi_1^* \det h_1^*)) \\ &\qquad \oplus \cdots \oplus H^q(\mathbb{P}(E_\ell^*), \Omega^p_{\mathbb{P}(E_\ell^*)} \otimes L_{E_\ell} \otimes \mathcal{I}(h_{L_{E_\ell}}^{1+r_\ell} \otimes \pi_\ell^* \det h_\ell^*)), \end{aligned}$$

*where $\pi_i : \mathbb{P}(E_i^*) \to X$ for $i = 1, \dots, \ell$.*

*Proof.* By the definition of multiplier submodule sheaves, we have

$$(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell)(h_1 \oplus \cdots \oplus h_\ell) = \mathcal{E}_1(h_1) \oplus \cdots \oplus \mathcal{E}_\ell(h_\ell).$$

Since cohomologies commute with direct sum of sheaves, we obtain

$$\begin{aligned} &H^q(X, W \otimes (\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell)(h_1 \oplus \cdots \oplus h_\ell)) \\ \simeq &H^q(X, W \otimes \mathcal{E}_1(h_1)) \oplus \cdots \oplus H^q(X, W \otimes \mathcal{E}_\ell(h_\ell)) \end{aligned}$$

and

$$\begin{aligned} &H^q(X, \Omega_X^p \otimes (\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell)(h_1 \oplus \cdots \oplus h_\ell)) \\ \simeq &H^q(X, \Omega_X^p \otimes \mathcal{E}_1(h_1)) \oplus \cdots \oplus H^q(X, \Omega_X^p \otimes \mathcal{E}_\ell(h_\ell)). \end{aligned}$$

By Theorem 1.3, we complete the proof. □

In [47], Wu proposed a problem whether a Nakano positive vector bundle always admits a strongly Nakano positive Hermitian metric. Here we construct a counterexample to disprove his problem.

**Example 4.5.** Let $L$ be a ample line bundle and $\underline{\mathbb{C}}$ be the trivial line bundle over a compact Riemann surface $\Sigma$. We consider the vector bundle $E := L \oplus \underline{\mathbb{C}}$. On the one hand, let $h$ be a positive metric on $L$. Then the metric $h \oplus \mathrm{Id}$ on $E$ is Nakano semi-positive.

On the other hand, $Y := \mathbb{P}(E^*) = \mathbb{P}(L^* \oplus \underline{\mathbb{C}})$ is the so-called *Hirzebruch-like surface* with $\ell := \deg(L^*) < 0$. More discussions on the Hirzebruch-like surfaces can be found in [16]. Subsequently, we do not distinguish divisors and line bundles for simplicity and abuse the notations of divisors and line bundles when no confusion can arise.

The Hirzebruch-like surface $Y = \mathbb{P}(L^* \oplus \underline{\mathbb{C}})$ can be viewed as the compactification of $L^*$ by adding a point at infinity to each fiber $L_x^*$. The union of these points is called the *infinity section*, denoted by $\Sigma_\infty$. Whereas the zero section of $L^*$ is regarded as a section of $\mathbb{P}(L^* \oplus \underline{\mathbb{C}})$ via the inclusion $L^* \subset \mathbb{P}(L^* \oplus \underline{\mathbb{C}})$, denoted by $\Sigma_0$.

It is well-known that

$$\mathcal{O}_{\mathbb{P}(E^*)}(1)=\mathcal{O}_Y(\Sigma_\infty).$$

Indeed, as stated in [22, Proposition 7.12], the projection of $L^*\oplus\underline{\mathbb{C}}$ to the trivial line bundle $\underline{\mathbb{C}}$ determines a section of $\mathcal{O}_{\mathbb{P}(E^*)}(1)$, whose zero divisor is $\Sigma_\infty$. Similarly, the projection of $L^*\oplus\underline{\mathbb{C}}$ to $L^*$ determines a section of $\mathcal{O}_{\mathbb{P}(E^*)}(1)\otimes\pi^*L^*$, and its zero divisor is $\Sigma_0$. This leads to the relation

$$\Sigma_0=\Sigma_\infty-\pi^*L.$$

Moreover, we also have

$$\Sigma_\infty\cdot\Sigma_\infty=-\ell,\quad \Sigma_0\cdot\Sigma_0=\ell,\quad \Sigma_0\cdot\Sigma_\infty=0.$$

Noticing that $\det(E)=\det(L)=L$, we get that

$$(r+1)\mathcal{O}_{\mathbb{P}(E^*)}(1)-\pi^*\det E=3\Sigma_\infty-\pi^*L=2\Sigma_\infty+\Sigma_0.$$

Hence we obtain that

$$((r+1)\mathcal{O}_{\mathbb{P}(E^*)}(1)-\pi^*\det E)\cdot\Sigma_0=\Sigma_0\cdot\Sigma_0=\ell<0.$$

Therefore, the line bundle $(r+1)\mathcal{O}_{\mathbb{P}(E^*)}(1)-\pi^*\det E$ can not be semi-positive. In other words, the vector bundle $E=L\oplus\underline{\mathbb{C}}$ is not strongly Nakano semi-positive.

By twisting the above vector bundle $E$ with a small ample line bundle, we also provide an example, which is Nakano positive but not strongly Nakano positive.

Concretely, consider the vector bundle $E:=L^N\oplus\underline{\mathbb{C}}$ for $N>1$ and

$$F:=L^{N+1}\oplus L=L\otimes E.$$

Then we have

$$\mathbb{P}(F^*)\simeq\mathbb{P}(E^*)\quad\text{ and }\quad\mathcal{O}_{\mathbb{P}(F^*)}(1)=\mathcal{O}_{\mathbb{P}(E^*)}(1)+\pi^*L.$$

One computes

$$\begin{aligned}3\mathcal{O}_{\mathbb{P}(F^*)}(1)-\pi^*\det(F)=&3\mathcal{O}_{\mathbb{P}(E^*)}(1)+3\pi^*L-\pi^*\det(L^{N+1}\oplus L)\\=&3\mathcal{O}_{\mathbb{P}(E^*)}(1)-(N-1)\pi^*L\\=&3\Sigma_\infty-\frac{N-1}{N}(\Sigma_\infty-\Sigma_0)\\=&(2+\frac{1}{N})\Sigma_\infty+(1-\frac{1}{N})\Sigma_0.\end{aligned}$$

Then we obtain that

$$(3\mathcal{O}_{\mathbb{P}(F^*)}(1)-\pi^*\det(F))\cdot\Sigma_0=(1-\frac{1}{N})N\deg(L^*)<0.$$

Thus, the vector bundle $F$ over $\Sigma$ is not strongly Nakano positive.

4.3. **Injectivity theorem.** In this subsection, we generalize Zhou–Zhu's injectivity theorem for pseudo-effective line bundles (see Theorem 2.16) to strongly Nakano semi-positive holomorphic vector bundles.

**Theorem 4.6.** *Let $(X,\omega)$ be a holomorphically convex Kähler manifold and $(E,h)\geq_{\mathrm{SNak}}0$. Assume that*

$$(E,h)\geq_{\mathrm{SNak}}b\sqrt{-1}\Theta_{(M,h_M)}$$

*for some $0<b<+\infty$, and*

$$\sqrt{-1}\Theta_{(M,h_M)}\geq-C\omega$$

*for some constant $C$. Then for any non-zero $s\in H^0(X,M)$ satisfying*

$$\sup_\Omega|s|_{h_M}<+\infty$$

*for every $\Omega \Subset X$, we obtain that the map*

$$H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes S^m\mathcal{E} \otimes \mathcal{M}(S^m h \otimes h_M))$$

*is injective for any $q \geq 0$.*

*Proof.* By setting $W = K_X$ in Theorem 1.3-(i) and observing that

$$K_{\mathbb{P}(E^*)/X} := K_{\mathbb{P}(E^*)} \otimes \pi^* K_X^{-1} = L_E^{-r} \otimes \pi^* \det E,$$

we derive the relationship

$$H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) \simeq H^q(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)} \otimes F \otimes \mathcal{I}(h_F)),$$

where $F = L_E^{m+r} \otimes \pi^* \det E^*$ and $h_F = h_L^{m+r} \otimes \pi^* \det h^*$.

Similarly, due to $\sqrt{-1}\Theta_{(M,h_M)} \geq -C\omega$, we obtain by Theorem 3.8 that

$$H^q(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)} \otimes F \otimes \pi^* M \otimes \mathcal{I}(h_F \otimes \pi^* h_M)) \simeq H^q(X, K_X \otimes S^m\mathcal{E} \otimes \mathcal{M}(S^m h \otimes h_M)).$$

In addition, since $(E,h) \geq_{\mathrm{SNak}} 0$ and $(E,h) \geq_{\mathrm{SNak}} \sqrt{-1}b\Theta_{(M,h_M)}$, it follows from Theorem 2.16 that $\pi^* s$ induces a injective morphism

$$H^q(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)} \otimes F \otimes \mathcal{I}(h_F)) \xrightarrow{\otimes \pi^* s} H^q(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)} \otimes F \otimes \pi^* M \otimes (h_F \otimes \pi^* h_M))$$

for any $q \geq 0$. Hence we conclude that the map

$$H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes S^m\mathcal{E} \otimes \mathcal{M}(S^m h \otimes h_M))$$

is injective for any $q \geq 0$. □

Following a similar process to that in Theorem 2.17, we also deduce that

**Corollary 4.7** (Torsion-freeness)**.** *Let $X$ be a Kähler manifold and $f$ be a proper holomorphic surjective morphism from $X$ to a complex analytic variety $Y$. Suppose that $(E,h) \geq_{\mathrm{SNak}} 0$ is a holomorphic vector bundle over $X$. Then $R^k f_*(K_X \otimes S^m\mathcal{E}(S^m h))$ is torsion-free for any $k \geq 0$. In particular, if $k > \dim X - \dim Y$, then*

$$R^k f_*(K_X \otimes S^m\mathcal{E}(S^m h)) = 0.$$

Let $M$ be a holomorphic line bundle over a Hermitian manifold $(X, \omega)$. Suppose that $h_M$ is a singular metric on $M$ satisfying that

$$\sqrt{-1}\Theta_{(M,h_M)} \geq -C\omega$$

for some constant $C$. We denote

$$H^0_{\mathrm{bdd},h_M}(X, M) := \{s \in H^0(X, M) \mid \sup_{\Omega} |s|_{h_M} < +\infty \text{ for any } \Omega \Subset X\}.$$

Assume that $\{h_k\}_{k=1}^{+\infty}$ is a sequence of singular metrics on holomorphic line bundle $M$. The *generalized Kodaira dimension* is defined as

$$\kappa_{\mathrm{bdd}}(M, \{h_k\}_{k=1}^{+\infty}) := \begin{cases} -\infty, & \text{if } H^0_{\mathrm{bdd},(h_k)^k}(X, M^k) = 0 \text{ for } k >> 0, \\ \sup\left\{ m \in \mathbb{Z} \,\middle|\, \limsup\limits_{k\to\infty} \frac{\dim H^0_{\mathrm{bdd},(h_k)^k}(X, M^k)}{k^m} > 0 \right\}, & \text{otherwise.} \end{cases}$$

We can derive the following vanishing result as a corollary of Theorem 4.6.

**Theorem 4.8.** *Let $(X, \omega)$ be a projective manifold and $(E,h) \geq_{\mathrm{SNak}} 0$ be a holomorphic vector bundle over $X$. Consider $M$ as a holomorphic line bundle equipped with a sequence of singular metrics $\{h_k\}_{k=1}^{+\infty}$ over $X$. Assume that the following two inequalities hold on $X$ in the sense of currents:*

(i) $(E,h)\geq_{\rm SNak}\varepsilon_k\sqrt{-1}\Theta_{(M,h_k)}$ *for any* $k\geq 1$ *and some sequence of positive numbers* $\{\varepsilon_k\}_{k=1}^{+\infty}$,

(ii) $k\sqrt{-1}\Theta_{(M,h_k)}\geq -C\omega$ *for any* $k\in\mathbb{N}$ *and some constant* $C>0$.

*Then we have*

$$H^q(X,K_X\otimes S^m\mathcal{E}(S^mh))=0$$

*for* $q>n-\kappa_{\rm bdd}(M,\{h_k\}_{k=1}^{+\infty})$.

*Proof.* Denote $\kappa_{\rm bdd}(M,\{h_k\}_{k=1}^{+\infty})$ by $\kappa$. There is nothing if $\kappa\leq 0$.

We only need to consider the case $\kappa>0$. By contradiction, suppose that there exists a non-zero cohomology class

$$\xi\in H^q(X,K_X\otimes S^m\mathcal{E}(S^mh))$$

for some $q>n-\kappa$.

For $k>0$, the following map $T_\xi$ induced by the tensor product with $\xi$

$$H^0_{{\rm bdd},(h_k)^k}(X,M^k)\longrightarrow H^q(X,K_X\otimes S^m\mathcal{E}\otimes\mathcal{M}^k(S^mh\otimes(h_k)^k))$$

is a linear map.

By assumptions that $(E,h)\geq_{\rm SNak}0$, $(E,h)\geq_{\rm SNak}\varepsilon_k\sqrt{-1}\Theta_{(M,h_k)}$ and $k\sqrt{-1}\Theta_{(M,h_k)}\geq -C\omega$, applying Theorem 4.6 for the above linear map $T_\xi$, we obtain that $T_\xi$ is injective. Hence, we derive

$$\dim H^0_{{\rm bdd},(h_k)^k}(X,M^k)\leq\dim H^q(X,K_X\otimes S^m\mathcal{E}\otimes\mathcal{M}^k(S^mh\otimes(h_k)^k)).$$

According to the following Lemma 4.9, one has

$$\dim H^q(X,K_X\otimes S^m\mathcal{E}\otimes\mathcal{M}^k(S^mh\otimes(h_k)^k))=O(k^{n-q})\qquad\text{as } k\to+\infty.$$

Then we have

$$\dim H^0_{{\rm bdd},(h_k)^k}(X,M^k)=O(k^{n-q})\qquad\text{as } k\to+\infty,$$

which implies that $\kappa\leq n-q$, contradicting the assumption $q>n-\kappa$. □

**Lemma 4.9.** *With the above notation, we conclude that*

$$\dim H^q(X,K_X\otimes S^mE\otimes M^k(S^mh\otimes(h_k)^k))=O(k^{n-q})\qquad\text{as } k\to+\infty.$$

*Proof.* We prove by induction on the dimension of manifolds. For simplicity, we denote

$$\mathcal{I}_k:=S^m\mathcal{E}\otimes\mathcal{M}^k(S^mh\otimes(h_k)^k).$$

For a given $k>0$, since $\mathcal{I}_k$ is a torsion-free coherent sheaf by Corollary 4.7, we can take a very ample smooth divisor $H$ so that $H$ does not contain any component of subvarieties defined by the associated primes of $\mathcal{I}_k$. Then we obtain a short exact sequence

$$0\to K_X\otimes\mathcal{I}_k\xrightarrow{\iota}K_X\otimes H\otimes\mathcal{I}_k\to(K_X\otimes H\otimes\mathcal{I}_k)|_H\to 0.$$

The above short exact sequence induces a long exact sequence

$$\cdots\to H^{q-1}(H,(K_X\otimes H\otimes\mathcal{I}_k)|_H)\to H^q(X,K_X\otimes\mathcal{I}_k)\to H^q(X,K_X\otimes H\otimes\mathcal{I}_k)\to\cdots.$$

Using Theorem 1.3, we deduce the following isomorphism

$$\begin{aligned}(14)\quad & H^q(X,K_X\otimes H\otimes\mathcal{I}_k)\\ &\simeq H^q(\mathbb{P}(E^*),K_{\mathbb{P}(E^*)}\otimes L_E^{m+r}\otimes\pi^*(\det E^*\otimes H\otimes M^k)\otimes\mathcal{I}(h_L^{m+r}\otimes\pi^*(\det h^*\otimes(h_k)^k))).\end{aligned}$$

Since $(E,h)\geq_{\rm SNak}0$ and $k\sqrt{-1}\Theta_{(M,h_k)}\geq -C\omega$, we can take $H$ very ample such that

$$\mathrm{nd}(L_E^{m+r}\otimes\pi^*(\det E^*\otimes H\otimes M^k),h_L^{m+r}\otimes\pi^*(\det h^*\otimes h_H\otimes(h_k)^k))=n+r-1,$$

where $h_H$ is a smooth Griffiths positive metric on $H$ satisfying

$$\sqrt{-1}\Theta_{(H,h_H)} \geq (C+1)\omega.$$

Then for $q, k > 0$, it follows from Theorem 2.11 and (14) that

$$H^q(X, K_X \otimes H \otimes \mathcal{I}_k) = 0.$$

Together with the above long exact sequence, we get that

$$\dim H^0(X, K_X \otimes \mathcal{I}_k) \leq O(k^n)$$

thanks to the Riemann–Roch formula. Additionly, we also obtain

$$\dim H^q(X, K_X \otimes \mathcal{I}_k) \leq \dim H^{q-1}(H, (K_X \otimes H \otimes \mathcal{I}_k)|_H) \leq O(k^{n-q}) \tag{15}$$

for $q > 0$ by induction and the adjunction formula $(K_X \otimes H)|_H = K_H$.

When $n = 1$, we can take $H = \sum_{j=1}^N p_j$ with $p_j \notin \mathrm{Supp}\ (S^m E \otimes M^k/\mathcal{I}_k)$ and $N$ independent of $k$. Furthermore, we have

$$\begin{aligned}\dim H^1(X, K_X \otimes \mathcal{I}_k) \leq & \dim H^0(H, (K_X \otimes H \otimes \mathcal{I}_k)|_H) \\ = & \sum_{j=1}^N \dim H^0(p_j, (K_X \otimes H \otimes \mathcal{I}_k)|_H) \\ = & \sum_{j=1}^N \dim H^0(p_j, (K_X \otimes H \otimes S^m E \otimes M^k)|_H) \\ = & N \ \mathrm{rank}(S^m E).\end{aligned}$$

□

Theorem 4.8 contains two distinctive cases.

The first case occurs when $h_k = h_M, k \in \mathbb{Z}_+$ for some fixed singular metric $h_M$ on $M$. Then the curvature assumptions (i) and (ii) are equivalent to the single assumption

$$(E, h) \geq_{\mathrm{SNak}} \varepsilon\sqrt{-1}\Theta_{(M,h_M)} \geq 0$$

for some positive number $\varepsilon$. If $h_M$ is further a smooth Griffiths positive metric, then

$$\kappa_{\mathrm{bdd}}(M, \{h_k\}_{k=1}^{+\infty}) = n.$$

Consequently, Theorem 4.8 transforms into a Nadel-type vanishing theorem on projective manifolds.

The second case arises when $M$ is numerically effective which means that there exists a sequence of smooth metrics $\{h_k\}_{k=1}^{+\infty}$ such that

$$\sqrt{-1}\Theta_{(M,h_k)} \geq -\frac{1}{k}\omega.$$

In this case, the curvature assumptions (ii) is automatically fulfilled, and $\kappa_{\mathrm{bdd}}(M, \{h_k\}_{k=1}^{+\infty})$ is equal to the usual Kodaira dimension of $M$.

4.4. **Vanishing theorems.** It is clear that a Kawamata–Viehweg–Nadel-type vanishing theorem for holomorphic vector bundles with strongly Nakano semi-positive metrics can be derived from Theorem 1.3 and Theorem 2.11.

**Corollary 4.10.** *Let $X$ be a compact Kähler manifold and $(E,h) \geq_{\mathrm{SNak}} 0$ be a holomorphic vector bundle over $X$. Then for any $m \geq 1$, we have*

$$H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) = 0$$

*for $q \geq n + r - \mathrm{nd}(L_E^{m+r} \otimes \pi^* \det E^*, h_L^{m+r} \otimes \pi^* \det h^*)$.*

In fact, $(E,h) \geq_{\mathrm{SNak}} 0$ implies that

$$\mathrm{nd}(L_E^{m+r} \otimes \pi^* \det E^*, h_L^{m+r} \otimes \pi^* \det h^*) \geq r-1.$$

Let us go into details. Take an open set $U \subset X$ such that $E|_U$ is trivial. Through a direct computation from (4) on $U \times \mathbb{P}^{r-1} \subset \mathbb{P}(E^*)$, we obtain

$$\sqrt{-1}\Theta_{(L_E,h_L)} \geq b\omega_{\mathrm{FS}},$$

where $\omega_{\mathrm{FS}}$ is regarded as the pull-back of the Fubini-Study metric on $\mathbb{P}^{r-1}$ and $b(z)$ is a non-negative continuous function satisfying that $b > 0$ outside a pluripolar set in $U$. Then there exists a positive constant $c > 0$ and a subset $B \subseteq U$ of positive measure satisfying

$$\sqrt{-1}\Theta_{(L_E,h_L)} \geq c\omega_{\mathrm{FS}} \text{ on } B \times \mathbb{P}^{r-1}.$$

By Demailly's approximation theorem, there exists a quasi-equisingular approximation $\{h_\varepsilon\}$ on $L_E$ over $\mathbb{P}(E^*)$ such that

$$\sqrt{-1}\Theta_{(L_E,h_\varepsilon)} \geq (b - \frac{1}{2}\varepsilon)\omega_{\mathrm{FS}}.$$

In particular, on $B \times \mathbb{P}^{r-1}$ we have

$$\sqrt{-1}\Theta_{(L_E,h_\varepsilon)} \geq \frac{c}{2}\omega_{\mathrm{FS}},$$

for sufficiently small $\varepsilon$. As $X$ is compact Kähler and so is $\mathbb{P}(E^*)$, by the definition of numerical dimension, we derive that

(i) $\mathrm{nd}(L_E, h_L) \geq r-1$ if $(E,h) \geq_{\mathrm{Grif}} 0$,
(ii) $\mathrm{nd}(L_E^{m+r} \otimes \pi^* \det E^*, h_L^{m+r} \otimes \pi^* \det h^*) \geq r-1$, if $(E,h) \geq_{\mathrm{SNak}} 0$.

Moreover, if $h$ is Griffiths positive, then we can get

$$\sqrt{-1}\Theta_{(L_E,h_L)} \geq a\pi^*\omega + b\omega_{\mathrm{FS}},$$

where $a(z)$ is a positive continuous function in $U$. Consequently, there exists a positive constant $c > 0$ and a subset $B \subseteq U$ of positive measure satisfying

$$\sqrt{-1}\Theta_{(L_E,h_L)} \geq c(\pi^*\omega + \omega_{\mathrm{FS}}) \text{ on } B \times \mathbb{P}^{r-1}.$$

Subsequently, a similar argument implies that

$$\mathrm{nd}(L_E, h_L) = n + r - 1.$$

Furthermore, if $(E,h) >_{\mathrm{SNak}} 0$, then

$$\mathrm{nd}(L_E^{m+r} \otimes \pi^* \det E^*, h_L^{m+r} \otimes \pi^* \det h^*) = n + r - 1.$$

**Corollary 4.11.** *Let $(E,h) >_{\mathrm{SNak}} 0$ be a holomorphic vector bundle of rank $r$ over a compact (possibly non-Kähler) manifold $X$ of dimension $n$, then for any $m \geq 1$, we have*

$$H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) = 0$$

*for $q \geq 1$.*

*Proof.* As $(E,h) >_{\mathrm{SNak}} 0$, we have

$$\int_{\mathbb{P}(E^*)} \left(\sqrt{-1}\Theta_{(L_E^{m+r}\otimes\pi^*\det E^*, h_L^{m+r}\otimes\pi^*\det h^*)}\right)^{n+r-1} > 0.$$

Thus, Lemma 2.13 implies that $\mathbb{P}(E^*)$ admits a Kähler modification $\mu : \widetilde{P} \to \mathbb{P}(E^*)$. Then applying Theorem 1.3 and Lemma 2.19, we obtain that

$$\begin{aligned}
&H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) \\
\simeq &H^q(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)} \otimes L_E^{m+r} \otimes \pi^*\det E^* \otimes \mathcal{I}(h_L^{m+r}\otimes\pi^*\det h^*)) \\
\simeq &H^q(\widetilde{P}, K_{\widetilde{P}} \otimes \mu^*(L_E^{m+r}\otimes\pi^*\det E^*) \otimes \mathcal{I}(\mu^*(h_L^{m+r}\otimes\pi^*\det h^*))).
\end{aligned}$$

Observing that

$$\int_{\widetilde{P}} \left(\sqrt{-1}\Theta_{(\mu^*(L_E^{m+r}\otimes\pi^*\det E^*), \mu^*(h_L^{m+r}\otimes\pi^*\det h^*))}\right)^{n+r-1} > 0,$$

we obtain by Theorem 2.11 that

$$H^q(X, K_X \otimes S^m\mathcal{E}(S^m h)) = 0,$$

for any $q \geq 1$. □

Similarly, one can show that

**Corollary 4.12** (Griffiths-type vanishing)**.** *Let $(E,h) >_{\mathrm{Grif}} 0$ be a holomorphic vector bundle of rank $r$ over a compact (possibly non-Kähler) manifold $X$ of dimension $n$, then for any $m \geq 1$, we have*

$$H^q(X, K_X \otimes S^m\mathcal{E} \otimes \det\mathcal{E}(S^m h \otimes \det h)) = 0$$

*for $q \geq 1$.*

The subsequent result serves as a generalization of Meng–Zhou's vanishing theorem for vector bundles with singular Hermitian metrics.

**Corollary 4.13.** *Let $X$ be a holomorphically convex Kähler manifold, and let $(E,h) \geq_{\mathrm{Grif}} 0$ be a holomorphic vector bundle on $X$. Suppose $h$ is smooth outside an analytic subset $Z$ and satisfies*

$$\left(\langle\sqrt{-1}\Theta(h^*)a, a\rangle(x)\right)^k \neq 0. \tag{16}$$

*for some point $x \in X\backslash Z$ and $a \in E_x^*$. Then we have*

$$H^q(X, K_X \otimes S^m\mathcal{E} \otimes \det\mathcal{E}(S^m h \otimes \det h)) = 0.$$

*for any $q \geq n-k+1$.*

*Proof.* Let $\{e_\alpha\}$ be a normal frame of $E$ at $x \in X\backslash Z$. The curvature of $h$ at $x$ is written as

$$\Theta_{(E,h)} = c_{ij\alpha\beta}dz^i \wedge d\overline{z}^j \otimes e_\alpha^* \otimes e_\beta,$$

satisfying $\overline{c}_{ij\alpha\beta} = c_{ji\beta\alpha}$. Additionally, the curvature of $h^*$ at $x$ is written as

$$\Theta_{(E,h^*)} = -c_{ij\beta\alpha}dz^i \wedge d\overline{z}^j \otimes e_\alpha \otimes e_\beta^*.$$

Write $a \in E_x^* \setminus \{0\}$ as $\sum a_\alpha e_\alpha^*$, then the curvature of $h_L$ at $(x,[a])$ is given by

$$\begin{aligned}
\Theta_{(L_E,h_L)}(x,[a]) &= \sum c_{ij\beta\alpha}\frac{a_\alpha\overline{a}_\beta}{|a|^2}dz^i\wedge d\overline{z}^j + \sum dw^\lambda \wedge d\overline{w}^\lambda \\
&= -\frac{\langle\Theta(h^*)a, a\rangle}{|a|^2} + \sum dw^\lambda\wedge d\overline{w}^\lambda. \qquad (17)
\end{aligned}$$

Using (16) and (17), we obtain that

$$\left(\sqrt{-1}\Theta_{(L_E,h_L)}(x,[a])\right)^{k+r-1} \neq 0.$$

Therefore, we conclude by Theorem 1.3 and Theorem 2.14 that

$$\begin{aligned}&H^q(X, K_X \otimes \mathcal{E} \otimes \det \mathcal{E}(h \otimes \det h))\\ =&H^q(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)} \otimes L_E^{r+1} \otimes \mathcal{I}(h_L^{r+1})) = 0.\end{aligned}$$

for any $q \geq n+r-1-(k+r-1)+1 = n-k+1$.

For $m \geq 2$, the proof follows a similar argument. □

4.5. **Singular holomorphic Morse inequalities for vector bundles.** Firstly, let us recall Bonavero's singular holomorphic Morse inequalities.

**Theorem 4.14** ([5]). *Let $Y$ be a compact complex manifold of dimension $n$ and let $L$ be a holomorphic line bundle on $Y$ equipped with a Hermitian metric $h_L$ with analytic singularities. Take $V$ as a holomorphic vector bundle on $Y$. Then for $0 \leq q \leq n$, we have*

$$h^q(Y, V \otimes L^m(h_L^m)) \leq \operatorname{rank}(V)\frac{m^n}{n!}\int_{Y(q)}(-1)^q(\sqrt{-1}\Theta_{(L,h_L)})^n + o(m^n), \tag{18}$$

*and*

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(Y, V \otimes L^m(h_L^m)) \leq \operatorname{rank}(V)\frac{m^n}{n!}\int_{Y(\leq q)}(-1)^q(\sqrt{-1}\Theta_{(L,h_L)})^n + o(m^n). \tag{19}$$

*where*

$$Y(q) := \{y \in Y | \sqrt{-1}\Theta_{(L,h_L)} \text{ has } q \text{ negative eigenvalues and } n-q \text{ positive eigenvalues}\},$$

*and*

$$Y(\leq q) := \bigcup_{j=0}^{q} Y(q).$$

According to Theorem 3.8, it is immediate to obtain a singular holomorphic Morse inequalities for vector bundle, as follows,

**Theorem 4.15.** *Let $X$ be a compact complex manifold of dimension $n$ and $(E,h)$ be a holomorphic vector bundle endowed with a singular Hermitian metric. Assume that there is a smooth function $\phi$ such that $(E, he^{-\phi}) \geq_{\mathrm{Grif}} 0$ and the induced metric $h_L$ on $L_E$ over $\mathbb{P}(E^*)$ has analytic singularities. Take $V$ as a holomorphic vector bundle on $X$. Then for $0 \leq q \leq n$, we have*

$$\begin{aligned}&h^q(X, V \otimes S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h))\\ \leq&\operatorname{rank}(V)\frac{m^{n+r-1}}{(n+r-1)!}\int_{\mathbb{P}(E^*)(q)}(-1)^q(\sqrt{-1}\Theta_{(L_E,h_L)})^{n+r-1} + o(m^{n+r-1}),\end{aligned}$$

*and*

$$\begin{aligned}&\sum_{j=0}^{q}(-1)^{q-j}h^j(X, V \otimes S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h))\\ \leq&\operatorname{rank}(V)\frac{m^{n+r-1}}{(n+r-1)!}\int_{\mathbb{P}(E^*)(\leq q)}(-1)^q(\sqrt{-1}\Theta_{(L_E,h_L)})^{n+r-1} + o(m^{n+r-1}).\end{aligned} \tag{20}$$

*where similarly*

$$\mathbb{P}(E^*)(q) := \{y \in \mathbb{P}(E^*) |\ \text{the curvature } \sqrt{-1}\Theta_{(L_E,h_L)} \text{ has } q \text{ negative eigenvalues}$$

*and $n+r-1-q$ positive eigenvalues*$\}$,

*and*

$$\mathbb{P}(E^*)(\leq q) := \bigcup_{j=0}^{q} \mathbb{P}(E^*)(q).$$

*Proof.* Since $h_L$ with analytic singularities, it follows from Theorem 3.8 and Theorem 4.14 that for any $0 \leq q \leq n$

$$\begin{aligned}
&h^q(X, V \otimes S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h)) \\
=&h^q(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)/X} \otimes \pi^* V \otimes L_E^{m+r} \otimes \mathcal{I}(h_L^{m+r})) \\
\leq&\mathrm{rank}(V) \frac{(m+r)^{n+r-1}}{(n+r-1)!} \int_{\mathbb{P}(E^*)(q)} (-1)^q (\sqrt{-1}\Theta_{(L_E,h_L)})^{n+r-1} + o(m^{n+r-1}),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{q} (-1)^{q-j} h^j(X, V \otimes S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h)) \\
=&\sum_{j=0}^{q} (-1)^{q-j} h^j(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)/X} \otimes \pi^* V \otimes L_E^{m+r} \otimes \mathcal{I}(h_L^{m+r})) \\
\leq&\mathrm{rank}(V) \frac{(m+r)^{n+r-1}}{(n+r-1)!} \int_{\mathbb{P}(E^*)(\leq q)} (-1)^q (\sqrt{-1}\Theta_{(L_E,h_L)})^{n+r-1} + o(m^{n+r-1}).
\end{aligned}$$

□

**Example 4.16.** Let $E$ be a trivial holomorphic vector bundle of rank $r$ over a domain $U \subset \mathbb{C}^n$. Let $\{F_k\}_{k \in \mathbb{Z}_+} \subset \mathcal{O}(U)$ be a sequence of holomorphic functions satisfying the following conditions:

(i) For any $K \Subset U$, there exists $C_K > 0$ and $N_K \in \mathbb{Z}_+$ such that on $K$

$$\sum_{k=1}^{+\infty} |F_k|^2 \leq C_K \sum_{k=1}^{N_K} |F_k|^2;$$

(ii) $\dim \mathrm{Span}_{k \in \mathbb{Z}_+} \{F_k(z)\} = r$ outside a proper analytic subset $A \subset U$.

For any fixed holomorphic frame $e = \{e_j\}_{j=1}^r$, one can define a singular metric $h^* =_{e^*} H$ on $E^*$ using the following matrix representation:

$$H = \sum_k (F_{k,i}\overline{F}_{k,j})_{r \times r} = \begin{pmatrix} \sum_k F_{k,1}\overline{F}_{k,1} & \cdots & \sum_k F_{k,1}\overline{F}_{k,r} \\ \vdots & \ddots & \vdots \\ \sum_k F_{k,r}\overline{F}_{k,1} & \cdots & \sum_k F_{k,r}\overline{F}_{k,r} \end{pmatrix}.$$

Then, for any local holomorphic section $G = \sum_{j=1}^r G_j e_j^*$ of $E^*$, we obtain that

$$|G|_{h^*}^2 = \sum_{i,j} (\sum_k F_{k,i}\overline{F}_{k,j}) G_i \overline{G}_j = \sum_k |\sum_i F_{k,i} G_i|^2$$

is a plurisubharmonic function. Therefore, $h^*$ is Griffiths semi-negative. Consequently, $(E, h) \geq_{\mathrm{Grif}} 0$. Additionally, according to (4), the induced metric $h_L$ on $L_E$ is represented by

$$\varphi_h(z, w) = \log \sum_k |\sum_i F_{k,i} w_i|^2.$$

Therefore, condition (1) implies that $h_L$ has analytic singularities.

**Example 4.17.** Let $E$ be a holomorphic vector bundle of rank $r$ over an n-dimensional compact Kähler manifold $X$. Assuming $\{F_k\}$ forms a basis of $H^0(X,E)$ such that outside a proper analytic subset $A \subset X$:

$$\dim \mathrm{Span}_k\{F_k(z)\} = r.$$

With a fixed holomorphic frame $e = \{e_j\}_{j=1}^r$, expressing $F_k$ as $F_k = \sum F_{k,j}e_j$, one can define a singular metric $h^* =_{e^*} H$ on $E^*$ represented by the following matrix:

$$H = \sum_k (F_{k,i}\overline{F}_{k,j})_{r\times r} = \begin{pmatrix} \sum_k F_{k,1}\overline{F}_{k,1} & \cdots & \sum_k F_{k,1}\overline{F}_{k,r} \\ \vdots & \ddots & \vdots \\ \sum_k F_{k,r}\overline{F}_{k,1} & \cdots & \sum_k F_{k,r}\overline{F}_{k,r} \end{pmatrix}.$$

Therefore, $(E,h) \geq_{\mathrm{Grif}} 0$, and the induced metric $h_L$ on $L_E$ has analytic singularities.

In addition, Theorem 4.14 yields an explicit formula about the asymptotic growth of global sections of pseudo-effective line bundle, which is equipped with a singular metric with analytic singularities.

Let $(M,h_M) \geq_{\mathrm{Grif}} 0$ be a holomorphic line bundle and $V$ be a holomorphic vector bundle on a compact Kähler manifold $Y$. Assume that $h_M$ has analytic singularities. On the one hand, taking $q=0$ in (18), we have

$$h^0(Y, V\otimes M^k \otimes \mathcal{I}(h_M^k)) \leq \mathrm{rank}(V)\frac{k^n}{n!}\int_Y (\sqrt{-1}\Theta_{(M,h_M)})^n + o(k^n). \tag{21}$$

On the other hand, taking $q=1$ in (19), we obtain that

$$h^0(Y, V\otimes M^k \otimes \mathcal{I}(h_M^k)) \geq \mathrm{rank}(V)\frac{k^n}{n!}\int_Y (\sqrt{-1}\Theta_{(M,h_M)})^n + o(k^n).$$

Thus, we conclude that

$$\lim_{k\to\infty}\frac{1}{k^n}h^0(Y, V\otimes M^k \otimes \mathcal{I}(h_M^k)) = \frac{\mathrm{rank}(V)}{n!}\int_Y (\sqrt{-1}\Theta_{(M,h_M)})^n. \tag{22}$$

Fix a smooth metric $h_0$ on $M$ and denote $\theta := \sqrt{-1}\Theta_{(M,h_0)}$. We denote by $\mathrm{Psh}(Y,\theta)$ the space of quasi-plurisubharmonic function $\varphi$ on $Y$ such that $\theta + \sqrt{-1}\partial\overline{\partial}\varphi \geq 0$ in the sense of currents. For any $\varphi \in \mathrm{Psh}(Y,\theta)$, we have $(M, h_0e^{-\varphi}) \geq_{\mathrm{Grif}} 0$. The so-called *$\mathcal{I}$-model envelope* is defined as

$$P[\varphi]_{\mathcal{I}}(z) := \sup\{\psi(z) \mid \psi \in \mathrm{Psh}(Y,\theta), \psi \leq 0, \mathcal{I}(k\psi) \subseteq \mathcal{I}(k\varphi), \forall\ k \in \mathbb{N}\} \tag{23}$$

for any $\varphi \in \mathrm{Psh}(Y,\theta)$. It is observed that $P[\varphi]_{\mathcal{I}} \in \mathrm{Psh}(Y,\theta)$. A quasi-plurisubharmonic function $\varphi \in \mathrm{Psh}(Y,\theta)$ is called *$\mathcal{I}$-model* if $P[\varphi]_{\mathcal{I}} = \varphi$.

Very recently, the authors in [12] extended the formula (22) to the case of $\mathcal{I}$-model quasi-plurisubharmonic function.

**Theorem 4.18** ([12])**.** *Let $(M, h_0e^{-\varphi}) \geq_{\mathrm{Grif}} 0$ be a Hermitian line bundle and $T$ be a holomorphic vector bundle of rank $s$ on a compact Kähler manifold $Y$. Then*

$$\lim_{k\to\infty}\frac{1}{k^n}h^0(Y, T\otimes M^k \otimes \mathcal{I}(k\varphi)) = \frac{s}{n!}\int_Y \left(\sqrt{-1}\Theta_{(M,h_0e^{-P[\varphi]_{\mathcal{I}}})}\right)^n, \tag{24}$$

*where $(\sqrt{-1}\Theta_{(M,h_0e^{-P[\varphi]_{\mathcal{I}}})})^n$ is the Monge-Ampère measure of $P[\varphi]_{\mathcal{I}}$ in the sense of* (6).

**Remark 4.19.** It follows the definition (23) of $\mathcal{I}$-envelope that $u \le P[u]_{\mathcal{I}} + C$ for some constant $C$. According to the monotonicity of Monge-Ampère masses [49, Theorem 1.1], one obtains that

$$\int_Y \left(\sqrt{-1}\Theta_{(M,h_0e^{-P[\varphi]_{\mathcal{I}}})}\right)^n \ge \int_Y \left(\sqrt{-1}\Theta_{(M,h_0e^{-\varphi})}\right)^n.$$

Take $(M, h_M) = (\det E, \det h)$ and $W = \mathcal{O}_X$ in Theorem 3.8 and Theorem 4.18. If $\mathbb{P}(E^*)$ is compact Kähler, then we obtain that

$$\begin{aligned}&\lim_{m\to\infty} \frac{1}{m^{n+r-1}} h^0(X, S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h))\\ =&\lim_{m\to\infty} \frac{1}{m^{n+r-1}} h^0(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)/X} \otimes L_E^{m+r}(h_L^{m+r}))\\ =&\frac{1}{(n+r-1)!}\int_{\mathbb{P}(E^*)} \left(\sqrt{-1}\Theta_{(L_E,P[h_L]_{\mathcal{I}})}\right)^{n+r-1}.\end{aligned}$$

Thus, we get an asymptotic formula for holomorphic vector bundles,

**Theorem 4.20.** *Let $X$ be a compact Kähler manifold and $(E,h) \ge_{\mathrm{Grif}} 0$. Then we have*

$$\begin{aligned}&\lim_{m\to\infty} \frac{1}{m^{n+r-1}} h^0(X, S^m\mathcal{E} \otimes \det \mathcal{E}(S^m h \otimes \det h))\\ =&\frac{1}{(n+r-1)!}\int_{\mathbb{P}(E^*)} \left(\sqrt{-1}\Theta_{(L_E,P[h_L]_{\mathcal{I}})}\right)^{n+r-1}.\end{aligned}$$

Lastly, We conclude this subsection by highlighting an asymptotic inequality for Griffiths positive vector bundles which should be analogous to that of big line bundles.

**Theorem 4.21.** *Let $(E,h) \ge_{\mathrm{Grif}} 0$ be a holomorphic vector bundle of rank $r$ over a compact (possibly non-Kähler) manifold $X$ of dimension $n$. Then we have*

$$\lim_{m\to\infty} \frac{1}{m^{n+r-1}} h^0(X, S^m\mathcal{E}(S^m h)) \ge \frac{1}{(n+r-1)!}\int_{\mathbb{P}(E^*)} \left(\sqrt{-1}\Theta_{(L_E,h_L)}\right)^{n+r-1}. \tag{25}$$

*Proof.* Without loss of generality, we may assume that

$$\int_{\mathbb{P}(E^*)} \left(\sqrt{-1}\Theta_{(L_E,h_L)}\right)^{n+r-1} > 0.$$

Thus, Lemma 2.13 implies that $\mathbb{P}(E^*)$ admits a Kähler modification $\mu : \widetilde{P} \to \mathbb{P}(E^*)$. Then using Lemma 2.19, we establish the relation

$$\begin{aligned}&H^q(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)} \otimes \pi^* \det E^* \otimes L_E^{m+r}(h_L^{m+r}))\\ \simeq&H^q(\widetilde{P}, K_{\widetilde{P}} \otimes \mu^*\pi^* \det E^* \otimes \mu^* L_E^{m+r} \otimes \mathcal{I}(\mu^* h_L^{m+r})).\end{aligned} \tag{26}$$

Additionally, it's apparent that

$$\int_{\widetilde{P}} \left(\sqrt{-1}\Theta_{(\mu^*L_E^{m+r},\mu^*h_L^{m+r})}\right)^{n+r-1} > 0.$$

Using Proposition 3.1, Theorem 4.20 together with the isomorphism (26), we obtain that

$$\begin{aligned}&\lim_{m\to\infty} \frac{1}{m^{n+r-1}} h^0(X, S^m\mathcal{E}(S^m h))\\ =&\lim_{m\to\infty} \frac{1}{m^{n+r-1}} h^0(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)/X} \otimes L_E^{m+r} \otimes \pi^* \det E^*(h_L^{m+r} \otimes \pi^* \det h^*))\\ \ge&\lim_{m\to\infty} \frac{1}{m^{n+r-1}} h^0(\mathbb{P}(E^*), K_{\mathbb{P}(E^*)/X} \otimes \pi^* \det E^* \otimes L_E^{m+r}(h_L^{m+r}))\end{aligned}$$

$$
\begin{aligned}
&= \lim_{m\to\infty} \frac{1}{m^{n+r-1}} h^0(\widetilde{P}, K_{\widetilde{P}/X} \otimes \mu^*\pi^* \det E^* \otimes \mu^* L_E^{m+r} \otimes \mathcal{I}(\mu^* h_L^{m+r}))\\
&\geq \frac{1}{(n+r-1)!} \int_{\widetilde{P}} \left(\sqrt{-1}\Theta_{(\mu^* L_E, \mu^* h_L)}\right)^{n+r-1}\\
&= \frac{1}{(n+r-1)!} \int_{\mathbb{P}(E^*)} \left(\sqrt{-1}\Theta_{(L_E, h_L)}\right)^{n+r-1},
\end{aligned}
$$

where the last inequality is due to Theorem 4.18 and Remark 4.19. □

**Remark 4.22.** If $h$ is Griffiths semi-positive on $X$ and merely positive on an open neighborhood, then the integral on the right-hand side of (25) is positive, which means that $H^0(X, S^m\mathcal{E}(S^m h))$ is not zero for $m >> 1$.

**Corollary 4.23.** *Let $(E,h) >_{\mathrm{Grif}} 0$ be a holomorphic vector bundle of rank $r$ over a compact Hermitian manifold $(X,\omega)$ of dimension $n$. Consider $\{x_1,\dots,x_N\}$ be a set of finite points and $s_j \geq 0$ for $j = 1,\dots N$. Then for $m >> 1$, there exists a nonzero global section $u$ of $S^mE$ vanishing at $x_j$ with order at least $s_j$, $j = 1,\dots,N$.*

*Proof.* By partition of unit, there exists a function $\theta$ on $X$ such that $\theta$ is smooth outside $\{x_1,\dots,x_N\}$ and equals to $2(n+s_j)\log|z-x_j|$ near $x_j$ for each $j$. Since $h$ is Griffiths positive, by definition one can show that there exists $m_0$ such that $e^{-\frac{1}{m_0}\theta}h$ is also Griffiths positive. Then for $m > m_0$ large enough, it follows from Theorem 4.21 that there is a nonzero global section $u$ of $S^mE$ with

$$
\int_X |u|^2_{S^m h} e^{-\frac{m}{m_0}\theta} dV_\omega < +\infty.
$$

As $h$ is locally bounded below by a continuous metric and $e^{-\theta}$ is equals to $\frac{1}{|z-x_j|^{2(n+s_j)}}$ near $x_j$ for each $j$, we obtain that $u$ vanishes at $x_j$ with order at least $s_j$, $j = 1,\dots,N$, due to $m > m_0$. □

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Department of Mathematics, University of Maryland, 4176 Campus Dr, College Park, MD 20742,USA; Tsinghua University, Department of Mathematics and Yau Mathematical Sciences Center, No 30, Shuangqing Road, Beijing 100084, China.

`yxliu238@umd.edu`

Beijing Institute of Mathematical Sciences and Applications, Beijing 101408, China; Department of Mathematics and Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China.

`liuzhuo@amss.ac.cn`

School of Mathematical Sciences, Peking University; Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, P. R. China

`yanghui@amss.ac.cn`

Institute of Mathematics, Academy of Mathematics and Systems Sciences, Beijing 100190, P. R. China

`xyzhou@math.ac.cn`